The Existence of an Equilibrium in a Lindahl Strategic Game

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This revision: October 2001

[Abstract]

This paper examines the existence of an equilibrium in a Lindahl strategic game of an economy with a public good and private goods in which agents manipulate their demand behaviors to improve their well-beings. In particular we shall establish the existence of an equilibrium under a certain set of assumptions which guarantee the positivity of consumption and price vectors of private goods and strategic parameters for private goods.

Keyword: public good, Lindahl equilibrium, strategic manipulation, existence
JEL Classification Numbers: C72, D50, D51, H40, H41

* The author is grateful to Yoshihiro Yoshida of Seikei University for his incisive comments on earlier versions of this paper which greatly improved the paper.

1. Introduction

In a related paper, we shall investigates properties of the set of allocations resulting from non-cooperative strategic interactions of agents who try to manipulate their demand behaviors to improve their well-beings in an economy with a public good (or public goods) and the Lindahl allocation mechanism. There we shall examine the effects of strategic externalities for the case of a public good economy. In particular, we would like to investigate what the size of strategic equilibrium allocations under the Lindahl mechanism would be and what the limiting set of strategic equilibrium allocations under this mechanism would be. But in order to examine these questions, we need to guarantee the existence of an equilibrium in our Lindahl strategic game in which consumption and price vectors of private goods and strategic parameters for private goods are all positive. The positivity of these variables is necessary to fruitful investigations of the manifold structure of the set of equilibrium allocations around a particular equilibrium whose existence would be assured in this paper.

In the next section, we give the model, definitions and assumptions used in this paper. In section 3, we point out several difficulties we have encountered in proving the existence. In section 4, we introduce a parameterized family of utility functions to reduce strategy spaces and show that this reduction does not cause any harm. Section 5 is devoted to the proof of the existence.

2. The Model, Definitions and Assumptions

We shall generally consider an economy with 1 public good, l private goods and T consumption agents. We abuse a notation using T for the number of agents as well as for the set of all agents.

2.1. Production of the public good

The public good is produced by the application of inputs of private goods and the technology is assumed to be represented by the production function $F: R^l_+ \to R_+$ denoted by $y = F(v)$. The production function is assumed to be continuous, strictly quasi-concave, $F(0) = 0$ and homogeneous of degree one. An input coefficient vector is denoted by $a_y \equiv v / y$. Given the price vector p of private goods, the unit cost function is defined as follows:

$$c(p) \equiv \min_{a_y} \left\{ p \cdot a_y \,\middle|\, F(a_y) \geq 1 \right\}$$

The minimizing vector of input coefficients as a function of prices of private goods will be denoted by: $a_y = a_y(p)$. Clearly $c(p) = p \cdot a_y(p)$. The profitability condition for the production of the public good is given as follows:

$$q - c(p) \leq 0, \text{ and}$$
$$y[q - c(p)] = 0$$

where $q \equiv \sum_{t \in T} q_t$ denotes the price of the public good with q_t being the contribution (or the cost share) of agent t.

2.2. Consumption agents

Consumption agent t is characterized by (u_t, ω_t) where $u_t : R_+^{l+1} \to R^*$ is his/her utility function and $\omega_t \in P^l$ with P indicating the set of strictly positive real numbers. The range R^* of utility functions is assumed to be the set of extended real numbers and thus possibly assume an infinite value at the boundary of the consumption set as in log-linear functions. We assume that the utility function is strictly quasi-concave, continuously differentiable on the positive orthant, $du_t = (d_x u_t, d_y u_t) \gg 0'_{l+1}$ and if $u_t(x_t, y) > u_t(\omega_t, 0)$, then $x_t \in P^l$. Note that we do not impose any assumption on the indispensability of a public good in consumption. This is because, in papers sequel to this, we would be interested in comparing an economy with a public good and one without. The budget map of agent t is defined as follows:

$$B_t(p, q_t) \equiv \left\{ (x_t, y) \in R_+^{l+1} \,\middle|\, p \cdot x_t + q_t \cdot y \leq p \cdot \omega_t \right\}$$

Let S_t be a parametric strategy space for agent t that is assumed to be a nonempty subset of a finite dimensional Euclidean space. We suppose that a given parameter $s_t \in S_t$ determines a strategic utility function $u_t(x_t, y, s_t)$ the agent uses. Strategic utility functions are assumed to be of the form $u_t(\bullet, s_t) : C(s_t) \to R^*$ where $C(s_t) \subseteq R_+^{l+1}$.

A strategic utility function in turn determines a strategic demand function $f_t : P^l \times R_+ \times S_t \to R_+^l$ for private goods and a cost share function $k_t : P^l \times R_+ \times S_t \to R_+$ for the public good as follows:

$$\begin{aligned}&[f_t(p, y, s_t), k_t(p, y, s_t)] \\ &\equiv \left\{(x_t, q_t) \middle| (x_t, y) = \arg\max \left\{u_t(x_t, y, s_t) \middle| (x_t, y) \in B_t(p, q_t) \cap C(s_t)\right\}\right\}\end{aligned}$$

2.3. Definitions of equilibria

Let us denote $S \equiv \prod_{t \in T} S_t$. A lindahl equilibrium given (a_y, c, s) and a consistent Lindahl equilibrium are defined as follows.

Definition 1. (a) $(p, y) \in P^l \times R_+$ is said to be a Lindahl equilibrium given $(c, a_y, s) \in R_+ \times R_+^l \times S$ if

(i) $\sum_{t \in T} \{ f_t(p, y, s_t) - \omega_t \} - a_y y = 0$, and

(ii) $\sum_{t \in T} k_t(p, y, s_t) - c = 0$.

(b) $(p, y) \in P^l \times R_+$ is said to be a consistent Lindahl equilibrium given $s \in S$ if, in addition to (i) and (ii) above,

(iii) $a_y = a_y(p)$ and $c = c(p)$

holds.

The set of Lindahl equilibria given (c, a_y, s) will be denoted by $L(c, a_y, s)$ and the set of consistent Lindahl equilibria given s will be denoted by $L(s)$.

Definition 2. $(p^*, y^*, x^*, s^*) \in P^l \times R_+ \times R_+^{lT} \times S$ is said to be a strategic Lindahl equilibrium if (i) $(p^*, y^*) \in L(s^*)$ and $x_t^* = f_t(p^*, y^*, s_t^*)$ for every $t \in T$, and (ii) for every

$t \in T$ and for every $s_t \in S_t$, if $(p,y) \in L(c^*, a_y^*, s^*/s_t)$, $c^* = c(p^*)$, $a_y^* = a_y(p^*)$, and $x_t = f_t(p,y,s_t)$, then $u_t(x_t, y) \le u_t(x_t^*, y^*)$.

We distinguish a Lindahl equilibrium given (c, a_y, s) and a consistent Lindahl equilibrium given s so that in a strategic Lindahl equilibrium, agents manipulate equilibria only through their demand, not through the production side of the public good by perceiving that their strategic behaviors do not affect the input coefficient vector and the unit cost of the public good. Therefore the strategic or incentive aspect of our model with respect to the public good is confined to that of a cost sharing game as in Bergstrom, Blume and Varian (1986),.

Suppose that budget's equality holds for every agent, i.e., $p \cdot x_t + q_t y = p \cdot \omega_t$ for every $t \in T$. Then summing over t and noting $[p \cdot a_y(p)]y = c(p)y$ yields,

$$p \cdot \sum_{t\in T}(x_t - \omega_t) + y\sum_{t\in T} q_t = p \cdot \sum_{t\in T}\left\{(x_t - \omega_t) + a_y(p)y\right\} + y\left\{\sum_{t\in T} q_t - c(p)\right\} = 0$$

which is the Walras law in our model. Given strategies (s/s_τ) of other agents and given the input coefficient vector, agent τ faces the excess supply of private goods available to him/her as follows.

$$x_\tau(p, y, a_y, s/s_\tau) \equiv \sum_{t\in T}\omega_t - \sum_{t\neq\tau} f_t(p, y, s_t) - a_y y$$

which will be called <u>the residual supply map</u> of private goods for agent τ.

3. Difficulties in Proving the Existence in Our Model

In this section, we would like to point out several difficulties we encounter in proving the existence of an equilibrium in our model.

(i) The Positivity of Variables on Private Goods

In a related paper, we would like to examine the manifold structure of Lindahl strategic equilibrium allocations. A standard procedure to investigate such a problem requires us to start from one particular equilibrium allocation and apply techniques available in differentiable topology and geometry to this equilibrium. In order to employ a differentiable approach, variables (consumption, price, and strategic variables) on private goods need to be strictly positive or in the interior of the space of these variables. On the other hand, we should not require the positivity on strategic variables on the public good because some agents may not want to pay for the public good. Moreover we may want to leave the possibility of the zero supply of the public good open as a result of strategic externalities yielding a strong under-supply tendency of the public good. This asymmetry in handling private goods and the public good requires some care in the proof.

(ii) Possible Non-convexity of the Residual Supply Map

The residual supply map is defined as follows.

$$X_\tau(a_y, s/s_\tau) \equiv \left\{(x_\tau, y) \in R_+^l \times R_+ \,\middle|\, \left(\exists p \in P^l\right)\left(x_\tau \le x_\tau(p, y, a_y, s/s_\tau)\right)\right\}.$$

The function $x_\tau(p, y, a_y, s/s_\tau)$ is the excess supply of private goods from agents other than agent τ. The Sonnenschein-Mantel-Debreu theorem shows that the excess demand (or supply) function aggregated over many agents do not possess more properties other than the continuity, the homogeneity of zero degree and the Walras law (see Mas-Colell, Whinston and Green [1995, pp. 598-606] for example). Therefore there is no guarantee that the residual supply map defined above is convex-valued. As in our previous paper (Otani [1996]), we avoid this problem by confining to a particular strategy subset in which the resulting residual supply map is convex-valued and moreover by proving that the restriction of strategies to this subset creates no problem.

When agents perceive the dependency of input coefficients and the unit cost of the public good on p, then each agent could try to exploit the public good production sector by trying to manipulate prices of private goods in input markets. Since $a_y(p)$ is not necessarily a convex function of p, the perception of $a_y(p)$ by each agent may result in the non-convexity of the residual supply. To avoid this type of non-convexity, we have introduced a Lindahl equilibrium given (c, a_y, s) in which agents accept (c, a_y) parametrically without recognizing its dependency on p.

(iii) The Feasibility of Individual Consumption

When other agents are willing to pay for a particular level of the public good, agent τ may not want to pay for the public good since the level desired by other agents may be sufficiently large for agent τ. Then agent τ will try to choose his/her desired consumption vector of private goods accepting (or constrained by) the level of the public good which other agents want. But we need some care in the level y of the public good that is set by other agents because a large y of the public good may result in the infeasibility of agent τ's consumption of private goods since the residual consumption function is defined as

$$x_\tau(p, y, a_y, s/s_\tau) \equiv \sum_{t\in T}\omega_t - \sum_{t\neq\tau} f_t(p, y, s_t) - a_y y .$$

4. Strategy Space Contraction and Strategic Lindahl Equilibria

Let us consider the following family of utility functions $u(\bullet, s): C(s) \to R^*$ with a vector of $2l$ strategic parameters $s \equiv (\alpha, \beta, \gamma, \delta) \in R_+^{2l}$.

$$u(x, y, s) \equiv \sum_{i=1}^{l-1}\alpha_i \ln(x_i - \beta_i) + x_l + \gamma \ln(y + \upsilon) + \delta y$$

in which $C(s) \equiv \left\{(x, y)\middle| x_i \geq \beta_i (i = 1, \cdots, l-1), x_l \geq 0, y \geq 0\right\}$, $\upsilon > 0$ is a fixed non-strategic parameter common to all agents and the role of υ is to enable us to admit $y = 0$. This utility function generates the following system of demand functions for private goods and the cost share function for the public good as follows;

$$f_i(p, y, s) \equiv (\alpha_i / p_i) + \beta_i \qquad (i = 1, 2, \cdots, l-1)$$

$$f_l(p, y, s) \equiv \sum_{i=1}^{l-1} p_i(\omega_i - \beta_i) + (\omega_l - \sum_{i=1}^{l-1}\alpha_i) - \gamma y/(y + \upsilon) - \delta y$$

$$k(y, s) \equiv \delta + \gamma/(y + \upsilon)$$

where p_i denotes the price of private good i relative to that of good l provided that $x_l = f_l(p, y, s) \geq 0$. Let $S_t^0 \equiv A_t \times B_t \times G_t \times D_t$ where

$$A_t \equiv R_+^{l-1}, B_t \equiv \left\{\beta \in R_t^{l-1} \middle| \beta_i \leq \omega_i \, (i = 1, \cdots, l-1)\right\}, G_t \equiv R_+, D_t \equiv R_+ .$$

We shall assume that the above family of utility functions is in the strategy set of each agent.

Assumption 1. For each $t \in T$, $S_t^0 \subseteq S_t$ holds.

When $s_t \in S_t^0$ for every $t \neq \tau$, then the residual supply map for private goods for agent τ is given as follows.

$$x_{\tau i}(p, y, a_y, s/s_\tau) \equiv \omega_{\tau i} + w_{\tau i} - a_{yi}y - [(a_{\tau i}/p_i) + b_{\tau i}], \quad (i = 1, \cdots, l-1),$$

$$x_{\tau l}(p, y, a_y, s/s_\tau) \equiv \omega_{\tau l} + a_\tau + g_\tau y/(y+\upsilon) + d_t y - a_{yl}y - \sum_{i=1}^{l-1} p_i(w_{\tau i} - b_{\tau i})$$

where $w_{\tau i} \equiv \sum_{t \neq \tau} \omega_{ti}$, $a_{\tau i} \equiv \sum_{t \neq \tau} \alpha_{ti}$, $b_{\tau i} \equiv \sum_{t \neq \tau} \beta_{ti}$, $g_\tau \equiv \sum_{t \neq \tau} \gamma_t$, $d_\tau \equiv \sum_{t \neq \tau} \delta_t$, and $a_\tau \equiv \sum_{i=1}^{l-1} a_{\tau i}$. Define

$$z_{\tau i}(p, s/s_\tau) \equiv (w_{\tau i} - b_{\tau i}) - (a_{\tau i}/p_i) \quad (i = 1, \cdots, l-1), \text{ and}$$
$$z_{\tau l}(p, s/s_\tau) \equiv a_\tau - \sum_{i=1}^{l-1} p_i(w_{\tau i} - b_{\tau i})$$

Then we have that

$$x_{\tau i}(p, y, a_y, s/s_\tau) = \omega_{\tau i} - a_{yi}y + z_{\tau i}(p, s/s_\tau) \qquad (i = 1, \cdots, l-1), \text{ and}$$

$$x_{\tau l}(p, y, a_y, s/s_\tau) \equiv \omega_{\tau l} - a_{yl}y + g_\tau y/(y+\upsilon) + d_\tau y + z_{\tau l}(p, s/s_\tau), \text{ or}$$

$$x_\tau(p, y, a_y, s/s_\tau) \equiv \omega_\tau - a_y y + [d_\tau y + g_\tau y/(y+\upsilon)]e_l + z_\tau(p, s/s_\tau)$$

with e_l denoting the l-th unit vector. Consider the following set-valued mapping

$$Z_\tau(s/s_\tau) \equiv \left\{ z_\tau \in R^l \,\middle|\, \left(\exists p \in P^l\right)\left(z_\tau \leq z_\tau(p, s/s_\tau)\right) \right\}$$

When $a_{\tau i}(w_{\tau i} - b_{\tau i}) > 0$ $(i = 1, \cdots, l-1)$, then the above can be rewritten as follows.

$$Z_\tau(s/s_\tau) = \left\{ z'_\tau \in R^l \,\middle|\, z'_\tau \leq z_\tau, z_{\tau l} = a_\tau - \sum_{i=1}^{l-1} a_{\tau i}(w_{\tau i} - b_{\tau i}) / \left\{(w_{\tau i} - b_{\tau i}) - z_{\tau i}\right\} \right\}.$$

The above set-valued mapping has the following properties as shown in Otani (1996).

Lemma 1. For each $\tau \in T$ and for each s/s_τ, (i) $Z_\tau(s/s_\tau)$ is a convex set bounded from above, and (ii) if $a_{\tau i}(w_{\tau i} - b_{\tau i}) > 0$ $(i = 1, \cdots, l-1)$, then $0_l \in Z_\tau(s/s_\tau)$ and $Z_\tau(s/s_\tau)$ is strictly convex.

Proof. To show $0_l \in Z_\tau(s/s_\tau)$, we set $z_{\tau i} = 0$ for $i = 1, \cdots, l-1$. Then we have that $z_{\tau l} = a_\tau - \sum_{i=1}^{l-1} a_{\tau i} = 0$. For the other properties, see Otani (1996) .□

The residual consumption map for agent τ is defined by

$$X_\tau(a_y, s/s_\tau) \equiv \left\{ (x_\tau, y) \in R_+^l \times R_+ \,\middle|\, x_\tau - \omega_\tau + a_y y - [d_\tau y + g_\tau y/(y+\upsilon)] e_l \in Z_\tau(s/s_\tau) \right\}.$$

The residual consumption map has properties similar to $Z_\tau(s/s_\tau)$ as follows.

Lemma 2. For each $\tau \in T$ and for each s/s_τ, (i) $X_\tau(a_y, s/s_\tau)$ is a convex set bounded from above, and (ii) if $a_{\tau i}(w_{\tau i} - b_{\tau i}) > 0$ ($i = 1, \cdots, l-1$), then $(\omega_\tau, 0) \in X_\tau(a_y, s/s_\tau)$ and the upper frontier of $X_\tau(a_y, s/s_\tau)$ is strictly convex.

Proof. To prove (i) and (ii), we note that the function $\psi(y) \equiv d_\tau y + g_\tau\, y/(y+\upsilon)$ is strictly concave in y .□

The residual cost share function $q_\tau : R_+ \times R_+ \times \prod_{t \neq \tau} S_t \to R$ for the public good will be defined as follows.

$$q_\tau(y, c, s/s_\tau) \equiv c - \sum_{t \neq \tau} k_t(y, s_t) = c - \sum_{t \neq \tau} \left(\delta_t + \gamma_t /(y+\upsilon) \right) = c - \frac{g_\tau}{y+\upsilon} - d_\tau$$

From the functional forms of demand functions for private goods and the cost share function for the public good, we can trivially assert the following.

Lemma 3. Given $(p, y, s_\tau) \in P^l \times R_+ \times S_\tau$, if $x_\tau = f_\tau(p, y, s_\tau) \gg 0_l$ and $q_\tau = k_\tau(y, s_\tau) \geq 0$, then there exists $s'_\tau \in S_\tau^0$ such that $x_\tau = f_\tau(p, y, s'_\tau)$ and $q_\tau = k_\tau(y, s'_\tau)$

with $(\alpha'_\tau, \beta'_\tau) \gg 0_{2(l-1)}$.

The next lemma asserts that a strategic Lindahl equilibrium with the strategy set S^0 indeed is a strategic Lindahl equilibrium with larger strategy set S. Of course the converse is not necessarily true.

Lemma 4. If $(p^*, y^*, x^*, s^*) \in P^l \times R_+ \times R_+^{lT} \times S^0$ is a strategic Lindahl equilibrium with respect to the strategy set S^0, then (p^*, y^*, x^*, s^*) is also a strategic Lindahl equilibrium with respect to the strategy set S.

Proof. Suppose that $s_\tau \in S_\tau$, and $(p, y) \in L(c^*, a_y^*, s^* / s_\tau)$ with $c^* = c(p^*)$ and $a_y^* = a_y(p^*)$. Let $x_\tau = f_\tau(p, y, s_\tau)$ and $q_\tau = k_\tau(p, y, s_\tau)$. Note that since $(p, y) \in L(c^*, a_y^*, s^* / s_\tau)$, we also have $x_\tau = x_\tau(p, y, a_y^*, s^* / s_\tau)$ and $q_\tau = q_\tau(y, c^*, s^* / s_\tau)$. If we have that $x_\tau \notin P^l$, then by the assumption on the utility function, $u_\tau(x_\tau, y) \le u_\tau(x_\tau^*, y^*)$ and the proof is over. Hence we may assume that $x_\tau \in P^l$. Clearly $q_\tau \ge 0$. By Lemma 3, there exists $s'_\tau \in S_\tau^0$ such that $x_\tau = f_\tau(p, y, s'_\tau)$ and $q_\tau = k_\tau(y, s'_\tau)$. Since (p^*, y^*, x^*, s^*) is a strategic Lindahl equilibrium with respect to S^0, we get $u_\tau(x_\tau, y) \le u_\tau(x_\tau^*, y^*)$. □

5. The Existence of a Strategic Lindahl Equilibrium

By Lemma 4 of the previous section, it suffices to restrict the strategy set to S^0 in proving the existence of a strategic Lindahl equilibrium. Also in proving the existence of a strategic equilibrium, the parameter δ_t in strategic utility functions will be redundant and hence we may assume that $\delta_t \equiv 0$. We note that the parameter δ_t will turn out to be useful in computing the dimension of strategic equilibrium consumption allocations as in the sequel paper Otani (2001 b). In this paper, we shall particularly be interested in proving the existence of equilibrium in which prices and consumption vectors are all positive, i.e., $(p^*, x^*) \in P^l \times P^{lT}$, and hence strategic parameters on private goods are also positive.

Theorem 1. There exists a strategic Lindahl equilibrium $(p^*, y^*, x^*, s^*) \in P^l \times R_+ \times R_+^{lT} \times S^0$ so that $(p^*, x^*) \in P^l \times P^{lT}$ and for every $t \in T$, $(\alpha_t^*, \beta_t^*) \in P^{2(l-1)}$ with $\beta_{ti}^* < \omega_{ti}$ $(i = 1, \cdots, l-1)$.

Proof. Given $\varepsilon > 0$ with $2\varepsilon < \min_{ti}\{\omega_{ti}\}$, choose $\lambda \in (0,1)$ and $M(\varepsilon) \ge \max_i \{(1/\varepsilon)\omega_i\}$ where $\omega \equiv \sum_{t\in T}\omega_t$. Then define

$$P^l(\varepsilon) \equiv \left\{ p \in P^l \middle| \left(p_l = 1\right) \wedge \left(\forall i = 1, \cdots, l-1\right)\left(\varepsilon \le p_i \le 1/\varepsilon\right)\right\},$$

$$A_t(\varepsilon) \equiv \left\{ \alpha_t \in P^{l-1} \middle| \left(\forall i = 1, \cdots, l-1\right)\left((1-\lambda)\varepsilon^2 \le \alpha_{ti} \le M(\varepsilon)\right)\right\},$$

$$B_t(\varepsilon) \equiv \left\{ \beta_t \in P^{l-1} \middle| \left(\forall i = 1, \cdots, l-1\right)\left(\lambda^2\varepsilon \le \beta_{ti} \le \omega_{ti} - \varepsilon\right)\right\},$$

$$\bar{y}(\varepsilon) \equiv \max\left\{ y \middle| a_y(p)y \le 2\omega, p \in P^l(\varepsilon)\right\},$$

$$\bar{c}(\varepsilon) \equiv \max\left\{ c \middle| c = p \cdot a_y(p), p \in P^l(\varepsilon)\right\},$$

$$\tilde{c}(p,\varepsilon) \equiv \max\{c(p), \varepsilon\}, \text{ and}$$

$$G_t(\varepsilon) \equiv [0, \bar{c}(\varepsilon)\left(\bar{y}(\varepsilon) + \upsilon\right)]$$

We modify the consumption set and strategy sets as follows.

$$R_+^{l+1}(\varepsilon) \equiv \left\{ (x,y) \in R_+^{l+1} \middle| \left(\forall i = 1, \cdots, l\right)\left(2\varepsilon \le x_i\right)\right\}$$

$$S_t^0(\varepsilon) \equiv A_t(\varepsilon) \times B_t(\varepsilon) \times G_t(\varepsilon), \text{ and}$$

$$S^0(\varepsilon) \equiv \prod_{t\in T} S_t^0(\varepsilon).$$

We proceed with the proof in several steps.

(i) Choice of strategic parameters

Define the residual supply map with free disposability for agent τ as follows.

$$X_\tau(a_y, s/s_\tau, \varepsilon) \equiv \left\{(x_\tau, y) \in R_+^l(\varepsilon) \times R_+ \middle| x_\tau - \omega_\tau + a_y y - [g_\tau y/(y+\upsilon)]e_l \in Z_\tau(s/s_\tau)\right\}$$

When y is large, then it may be impossible to find x_τ so that $(x_\tau, y) \in X_\tau(a_y, s/s_\tau, \varepsilon)$. So let us define the largest y so that such an x_τ can be found, i.e.,

$$\overline{y}_\tau(a_y, s/s_\tau, \varepsilon) \equiv \max\left\{y \middle| \left(\exists x_\tau \in R_+^l(\varepsilon)\right)\left((x_\tau, y) \in X_\tau(a_y, s/s_\tau, \varepsilon)\right)\right\}.$$

Using the residual cost share function, we may define the level of the public good which agents other than τ are willing to provide by setting $q_\tau(p, y, s/s_\tau) = 0$, i.e., $y = \left(\sum_{t\neq\tau}\gamma_t \big/ c\right) - \upsilon = (g_\tau/c) - \upsilon$ provided that $0 < c \leq g_\tau/\upsilon$. To obtain the nonnegative supply with the well-defined optimization, we define for $c \in [\varepsilon, \overline{c}(\varepsilon)]$,

$$\tilde{y}_\tau(c, a_y, s/s_\tau, \varepsilon) \equiv \min\left\{\max\left\{(g_\tau/c) - \upsilon, 0\right\}, \overline{y}_\tau(a_y, s/s_\tau, \varepsilon)\right\}$$

and call this as the residual supply map of the public good for agent τ. Then the optimization problem of agent τ can be formulated as follows:

$$\max_{(x_\tau, y)}\left\{u_\tau(x_\tau, y) \middle| (x_\tau, y) \in X_\tau(a_y, s/s_\tau, \varepsilon), y \geq \tilde{y}_\tau(c, a_y, s/s_\tau, \varepsilon)\right\}.$$

We may consider two optimization problems for agent τ depending on whether the public good constraint $y \geq \tilde{y}_\tau(c, a_y, s/s_\tau, \varepsilon)$ is unbinding or binding. First we consider the case in which the public good constraint is unbinding, i.e.,

$$\max_{(x_\tau, y)}\left\{u_\tau(x_\tau, y) \middle| (x_\tau, y) \in X_\tau(a_y, s/s_\tau, \varepsilon)\right\}$$

The above optimization is well-defined since $(\omega_\tau, 0) \in X_\tau(a_y, s/s_\tau, \varepsilon)$. The solution pair of private and public goods as functions of $(c, a_y, s/s_\tau)$ will be denoted by $[x_\tau(c, a_y, s/s_\tau, \varepsilon), y_\tau(c, a_y, s/s_\tau, \varepsilon)]$. Clearly $y_\tau(c, a_y, s/s_\tau, \varepsilon) \leq \overline{y}_\tau(a_y, s/s_\tau, \varepsilon)$.

We consider another optimization constrained by the level of the public good y, namely given $y \in [0, \bar{y}_\tau(a_y, s/s_\tau, \varepsilon)]$,

$$\max_{x_\tau} \left\{ u_\tau(x_\tau, y) \middle| (x_\tau, y) \in X_\tau(a_y, s/s_\tau, \varepsilon) \right\}.$$

The solution to this y-constrained optimization will be denoted by $x_\tau(y, a_y, s/s_\tau, \varepsilon)$.

When the residual supply for the public good is not sufficiently large, $\tilde{y}_\tau(c, a_y, s/s_\tau, \varepsilon) \le y_\tau(c, a_y, s/s_\tau, \varepsilon)$, then agent τ will be willing to pay his share $q_\tau = c - \left[g_\tau \middle/ \left\{ y_\tau(c, a_y, s/s_\tau, \varepsilon) + \upsilon \right\} \right]$ and agent τ sets his/her strategic parameter on the public good as $\gamma_\tau(c, a_y, s/s_\tau) \equiv c[\upsilon + y_\tau(c, a_y, s/s_\tau, \varepsilon)] - g_\tau$ to increase the level of the public good supply to his/her desirable level $y_\tau(c, a_y, s/s_\tau, \varepsilon)$. The corresponding consumption of private goods for agent τ will be unconstrained by y given by $x_\tau(a_y, s/s_\tau, \varepsilon)$. When the residual supply is sufficiently large, $y_\tau(a_y, s/s_\tau, \varepsilon) \le \tilde{y}_\tau(c, a_y, s/s_\tau, \varepsilon)$, agent τ will not pay his share and accept the residual supply $\tilde{y}_\tau(c, a_y, s/s_\tau, \varepsilon)$ as a constraint in deciding his/her choice of private goods. In this case $\gamma_\tau(c, a_y, s/s_\tau, \varepsilon) = 0$, and the consumption of private goods is given by $x_\tau(c, a_y, s/s_\tau, \varepsilon) \equiv x_\tau[\tilde{y}_\tau(c, a_y, s/s_\tau, \varepsilon), a_y, s/s_\tau, \varepsilon]$ constrained by the level of the public good $\tilde{y}_\tau(c, a_y, s/s_\tau, \varepsilon)$.

Combining the above two cases, we shall write the demand mapping for private and public goods as $[x_\tau, y_\tau] = [\hat{x}_\tau(c, a_y, s/s_\tau, \varepsilon), \hat{y}_\tau(c, a_y, s/s_\tau, \varepsilon)]$. Define

$$\hat{z}_\tau(c, a_y, s/s_\tau, \varepsilon) \equiv x_\tau - \omega_\tau + a_y y_\tau - [g_\tau y_\tau / (y_\tau + \upsilon)] e_l$$

where $[x_\tau, y_\tau] = [\hat{x}_\tau(c, a_y, s/s_\tau, \varepsilon), \hat{y}_\tau(c, a_y, s/s_\tau, \varepsilon)]$. The private goods in the above ought to be supplied by other agents as the residual supply of private goods and the corresponding price vector must satisfy that $z_{\tau i}(p, s/s_\tau) = \hat{z}_{\tau i}(c, a_y, s/s_\tau, \varepsilon)$ where $z_{\tau i}(p, s/s_\tau) = (w_{\tau i} - b_{\tau i}) - (a_{\tau i}/p_i)$. This defines a price mapping $\eta_\tau(c, a_y, s/s_\tau, \varepsilon)$ for agent τ where

$$\eta_{\tau i}(c, a_y, s/s_\tau, \varepsilon) \equiv a_{\tau i} \Big/ \left\{(w_{\tau i} - b_{\tau i}) - \hat{z}_{\tau i}(c, a_y, s/s_\tau, \varepsilon)\right\} \quad (i = 1, \cdots, l-1)$$

and $\eta_{\tau l}(c, a_y s/s_\tau, \varepsilon) \equiv 1$. Since there is no guarantee that values of the above price mapping stay in $P^l(\varepsilon)$, we modify this price mapping for agent τ to $p_\tau(c, a_y, s/s_\tau, \varepsilon)$ so that it lies in $P^l(\varepsilon)$ as follows:

$$p_{\tau i}(c, a_y, s/s_\tau, \varepsilon) \equiv \max\left\{\varepsilon, \min\left\{\eta_{\tau i}(c, a_y, s/s_\tau, \varepsilon), 1/\varepsilon\right\}\right\} \quad (i = 1, \cdots, l-1), \text{ and}$$

$$p_{\tau l}(c, a_y, s/s_\tau, \varepsilon) \equiv \eta_{\tau l}(c, a_y, s/s_\tau, \varepsilon) = 1.$$

Given $\hat{x}_\tau(c, a_y, s/s_\tau, \varepsilon)$ and $p_\tau = p_\tau(c, a_y, s/s_\tau, \varepsilon)$, as in Otani (1996), we may obtain strategic parameters for private goods as $[\alpha_\tau(c, a_y, s/s_\tau, \varepsilon), \beta(c, a_y, s/s_\tau, \varepsilon)]$. Then the strategy mapping for agent τ is defined by

$$\tilde{\psi}_\tau(c, a_y, s/s_\tau, \varepsilon) \equiv [\alpha_\tau(c, a_y, s/s_\tau, \varepsilon), \beta_\tau(c, a_y, s/s_\tau, \varepsilon), \gamma_\tau(c, a_y, s/s_\tau, \varepsilon)].$$

(ii) Fixed point argument

Now we shall define the economy-wide mapping for the public good and the economy-wide price mapping as follows. First the economy-wide mapping for the public good is given by

$$y(c, a_y, s, \varepsilon) \equiv \max\left\{\left(1/c\right)\left(\sum_{t \in T} \gamma_t\right) - \upsilon, 0\right\}.$$

Now define

$$h_i(c, a_y, s, \varepsilon) \equiv \sum_{t\in T}(\omega_{ti} - \beta_{ti}) - a_{yi} y(c, a_y, s, \varepsilon).$$

Then the economy-wide price mapping for private good i $(i = 1, \cdots, l-1)$ is given by

$$\tilde{\varphi}_i(c, a_y, s, \varepsilon) \equiv \min\left\{1/\varepsilon, \max\left\{\sum_{t\in T}\alpha_{ti} \Big/ h_i(c, a_y, s, \varepsilon), \varepsilon\right\}\right\}$$

and $\tilde{\varphi}_l(c, a_y, s, \varepsilon) \equiv 1$. Substituting $(c, a_y) = \left(\tilde{c}(p, \varepsilon), a_y(p)\right)$ into the price mapping $p(c, a_y, s/s_\tau, \varepsilon)$ for private goods and the strategy mapping $\tilde{\psi}_\tau(c, a_y, s/s_\tau, \varepsilon)$ for each agent, we obtain that

$$\varphi(p, s, \varepsilon) \equiv \tilde{\varphi}(\tilde{c}(p), a_y(p), s, \varepsilon)$$

$$\psi_t(p, s, \varepsilon) \equiv \tilde{\psi}_t(\tilde{c}(p, \varepsilon), a_y(p), s/s_t, \varepsilon) \text{ for each } t \in T.$$

Finally we obtain a mapping $F_\varepsilon : P^l(\varepsilon) \times S^0(\varepsilon) \to P^l(\varepsilon) \times S^0(\varepsilon)$ by

$$F_\varepsilon(p, s) \equiv [\varphi(p, s, \varepsilon), \psi_1(p, s, \varepsilon), \cdots, \psi_T(p, s, \varepsilon)].$$

By our construction, the above mapping is clearly a continuous mapping from $P^l(\varepsilon) \times S^0(\varepsilon)$ to itself. Therefore, by Brouwer's fixed point theorem, there exists a fixed point $(p(\varepsilon), s(\varepsilon)) \in F_\varepsilon(p(\varepsilon), s(\varepsilon))$. For each $\varepsilon > 0$, let us define

$$(c(\varepsilon), a_y(\varepsilon)) \equiv [\tilde{c}(p(\varepsilon), \varepsilon), a_y(p(\varepsilon))], \quad \eta_t(\varepsilon) \equiv \eta_t[c(\varepsilon), a_y(\varepsilon), s(\varepsilon)/s_t(\varepsilon), \varepsilon],$$

$$p_t(\varepsilon) \equiv p_t[c(\varepsilon), a_y(\varepsilon), s(\varepsilon)/s_t(\varepsilon), \varepsilon], \quad y_t(\varepsilon) \equiv y_t[c(\varepsilon), a_y(\varepsilon), s(\varepsilon)/s_t(\varepsilon), \varepsilon],$$

$$y(\varepsilon) \equiv y[c(\varepsilon), a_y(\varepsilon), s(\varepsilon), \varepsilon], \text{ and } x_{ti}(\varepsilon) \equiv [\alpha_{ti}(\varepsilon)/p_{ti}(\varepsilon)] + \beta_{ti}(\varepsilon) \quad \text{for}$$

$$(i = 1, \cdots, l-1).$$

(iii) Equilibrium properties

As in Otani (1996), we can show that for sufficiently small $\varepsilon > 0$, we must have that $p_t(\varepsilon) = \eta_t(\varepsilon)$ for every $t \in T$. Thus when $p_t(\varepsilon) = \eta_t(\varepsilon)$, we have that

$$(w_{ti} - b_{ti}) - \hat{z}_{ti}[c(\varepsilon), a_y(\varepsilon), s(\varepsilon)/s_t(\varepsilon)] = a_{ti}(\varepsilon)/p_{ti}(\varepsilon).$$

Noting that $\hat{z}_{ti}[c(\varepsilon), a_y(\varepsilon), s(\varepsilon)/s_t(\varepsilon)] = x_{ti}(\varepsilon) - \omega_{ti} + a_y(\varepsilon) y_t(\varepsilon)$ for $i \neq l$, we get

$$x_{ti}(\varepsilon) = [\alpha_{ti}(\varepsilon)/p_{ti}(\varepsilon)] + \beta_{ti}(\varepsilon) = \omega_{ti} + [w_{ti} - b_{ti}(\varepsilon)] - [a_{ti}(\varepsilon)/p_{ti}(\varepsilon)] - a_{yi}(\varepsilon)y_t(\varepsilon)$$

Thus we obtain

$$p_{ti}(\varepsilon) = a_i(\varepsilon)[\omega_i - b_i(\varepsilon) - a_{yi}(\varepsilon)y_t(\varepsilon)]^{-1}$$

where $\omega_i \equiv \sum_{t\in T}\omega_{ti}$, $a_i(\varepsilon) \equiv \sum_{t\in T}\alpha_{ti}(\varepsilon)$, and $b_i(\varepsilon) \equiv \sum_{t\in T}\beta_{ti}(\varepsilon)$. When $y(\varepsilon)=0$, then our model reduces to the economy with private goods and our proof in Otani [1996] applies. Thus we may assume that $y(\varepsilon)>0$. Then

$$y(\varepsilon) = \left(1/c\right)\left(\sum_{t\in T}\gamma_t(\varepsilon)\right) - \upsilon$$

and there exists at least one t for which $\gamma_t(\varepsilon)>0$ and for this agent, the consumption of private goods is not $y(\varepsilon)$-constrained. For every agent τ with $\gamma_\tau(\varepsilon)>0$,

$$\gamma_\tau(\varepsilon) = c[\upsilon + y_\tau(\varepsilon)] - g_\tau(\varepsilon)$$

and thus

$$y_\tau(\varepsilon) = \left(1/c(\varepsilon)\right)\left(\sum_t \gamma_t(\varepsilon)\right) - \upsilon = y(\varepsilon).$$

Since all $y(\varepsilon)$-constrained agents are constrained by $y(\varepsilon)$, we can assert that for every agent t,

$$p_{ti}(\varepsilon) = a_i(\varepsilon)[\omega_i - b_i(\varepsilon) - a_{yi}(\varepsilon)y(\varepsilon)]^{-1} = p_i(\varepsilon), \text{ or}$$

$$\omega_i - b_i(\varepsilon) - a_{yi}(\varepsilon)y(\varepsilon) = a_i(\varepsilon)/p_i(\varepsilon)$$

which implies the feasibility condition

$$\left[\left(a_i(\varepsilon)/p_i(\varepsilon)\right) + b_i(\varepsilon)\right] + a_{yi}(\varepsilon)y(\varepsilon) = \sum_{t\in T} x_{ti}(\varepsilon) + a_{yi}(\varepsilon)y(\varepsilon) = \omega_i \quad (i = 1,\cdots,l-1)$$

for sufficiently small $\varepsilon>0$. Thus we may call $(p(\varepsilon), y(\varepsilon), x(\varepsilon), s(\varepsilon))$ an ε-equilibrium.

Although it takes fairly involved arguments as in Otani [1996, pp.223-224], we can show that $(p(\varepsilon), y(\varepsilon), x(\varepsilon), s(\varepsilon))$ has a limit point denoted $(p^*, y^*, x^*, s^*) \in P^l \times R_+ \times R_+^{lT} \times S^0$ and this limit point is a strategic Lindahl equilibrium such that $(p^*, x^*) \in P^l \times P^{lT}$ and for every $t\in T$, $(\alpha_t^*, \beta_t^*) \in P^{2(l-1)}$. □

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