The Economic Structure of the Seigniorial System in the Steady State:

A Mathematical Formulation

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[Abstract]

In the great number of pre-modern societies "the rule over the land" was well established; that is to say, kings, noblemen, military officers, government officials, local magnates, high clergymen and other seigniors exerted their influence over larger or smaller areas and brought under their authority a number of peasants living there. In this paper, constructing a mathematical model on the assumption that the system of the social division of labor between agriculture and industry is well established, we explore what brings about "the rule over the land" in the pre-modern society.

  First, we have demonstrated that, with highly volatile agricultural production, the resource allocation which comes about as a result of free action of both peasants and craftsmen does not allow the system of the social division of labor between agriculture and industry to continue. Second, it follows that, for its continuance the peasants' and the craftsmen's choices of consumption goods should be restricted by the seignior, who, as the third economic agent, controls their free action and brings under his authority a number of peasants in particular.

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1. Introduction

The societies preceding the Industrial Revolution or industrialization are gener-
ally called pre-modern societies. In the great number of pre-modern societies "the
rule over the land" was well established; that is to say, kings, noblemen, military
officers, government officials, local magnates, high clergymen and other seigniors
exerted their influence over larger or smaller areas and brought under their au-
thority a number of peasants living there. Then, why was "the rule over the land"
referred to above established? To solve the problem, the author so far has carried
out a series of investigations into the economic structure of the seigniorial sys-
tem[1]. In spite of, however, a series of investigations, the problem has not been
sufficiently studied yet. In this paper we will make a further inquiry about several
points which have still remained unsettled in the preceding studies.

To begin with, we will make clear in what context the problem has been raised.
At first glance the question of what caused "the rule over the land" may seem to
ask why each seignior, sovereign included, held sway over his tenants and exacted
farm products from them in his individual case. But, kings, noblemen, military
officers, government officials, local magnates, high clergymen and others could be
a seignior, who brought neighboring peasants under his authority and collected
their farm products, provided that he, as a seignior, fulfilled a definite economic
function. Each seignior, exercising part of his economic function, exerted his in-
fluence over larger or smaller areas and kept a number of peasants at his command.
"The rule over the land", in other word, is part of the economic function of the lord.
If that is the case, in so far as we take for grant the presence of the seignior as an
agent in charge of the particular economic function, "the rule over the land", as
part of this function, comes to be self-evident, and the problem at the beginning of

this paper no longer deserves consideration. Therefore, the core of the problem which truly demands a serious consideration lies in, without taking for granted the presence of the lord, bringing his raison d'être into question. That is to say, our object in this paper is to explore why there was an economic agent, called seignior, in charge of particular economic functions.

Then, what economic functions was the seignior responsible for? From observations of actual pre-modern societies in which "the rule over the land" was maintained, we have known that seigniors, along with peasants and craftsmen, played no negligible role in the social division of labor in these societies. While peasants chiefly take on agricultural production and craftsmen chiefly undertake industrial production, seigniors intervene in the exchange of products between them. The economic function of the lord consists in the intervention in the exchange between agricultural and industrial products[2].

Peasants, craftsmen, and seigniors perform their particular economic functions, and the social division of labor is established in the pre-modern society, more precisely in the pre-modern society in which "the rule over the land" is noticed. Like the peasant and the craftsman, the seignior also is one of constituent elements of the social division of labor. Therefore, the presence of the seignior is inseparable from this social division of labor; we can not account for this social division of labor without referring to the presence of the seignior, nor, conversely, can the seignior be present without making the social division of labor. In the end the enquiry into the raison d'être of the seignior is nothing but studying the contributory causes of the social division of labor.

The first question of why "the rule over the land" was established is virtually that of inquiring into the raison d'être of the seignior, and, ultimately, arrives at

that of asking for the contributory causes of the social division of labor in the pre-modern society. We find that the seemingly naïve question is part of the larger and more profound problem concerning the existence of the economic structure of the seigniorial system.

In a series of theoretical studies on the economic structure of the seigniorial system, we have shed considerable light on the subject and so far succeeded in having several results. Let's take a brief look at the result up to the present. The social division of labor is, to begin with, the division of roles between peasants and craftsmen, both taking part in productive activities, and at the same time the division of roles between those who are taking part in productive activities and those who are not. Therefore, the problem with the formation of the social division of labor can be divided into two parts. First, what causes the social division of labor between peasants and craftsmen? The former part of the problem asks for the contributory causes of the social division of labor between agriculture and industry. As for this part, we have already shown that job skills of a high order required for some areas of industrial production result in the formation of the social division of labor between agriculture and industry[3]. Second, on the premise that the system of the social division of labor between agriculture and industry is established, why is a particular economic role assigned to the seignior under the system? The latter part of the problem asks for the raison d'être of the seignior under the system of the social division of labor between agriculture and industry. Regarding this part, we have pointed out that if we are determined to preserve the system under the circumstances of highly volatile agricultural production it is inevitable that the seignior intervenes in the process of exchange between agricultural and industrial products[4].

Nevertheless, as for the latter part of the problem, in spite of our studies up to the present, our analysis is still far from being thorough. Thus, in this paper, we will show that, taking it for granted that the system of the social division of labor between agriculture and industry prevails, the presence of the seignior is indispensable for the persistence of the system in a mathematical model, and will formulate in rigorous terms the analysis so far conducted.

The formation of the social division of labor between agriculture and industry requires that production technology take on certain properties. In section 2 in this paper, we set out a number of conditions of productive activities and private consumption which are necessary for the system of the social division of labor between agriculture and industry to be established. In section 3, we will make clear the requirements for the continuance of the system. Anyway, can this system of the social division of labor be sustained by the sheer force of peasants and craftsmen? In section 4 and section 5 in the following, we will show that if we decide to preserve the system an intervention of a third party different from either the peasant or the craftsman is inevitable. Finally, in the section 6, we will consider what will become of the resource allocation if the seignior as a third party intervenes in the system of the social division of labor between agriculture and industry.

2. The peasant and the craftsman

The social production in the pre-modern society comprises overwhelmingly dominant agricultural production and a relatively small scale of industrial production. To begin with, let's set out some conditions of the social production clearly. For the sake of simplicity, we assume that the community produces a single sort of agri-

cultural products and a single sort of industrial products. As for agricultural production, using $\overline{D}$ units of industrial products including farming tools, anyone can bring forth X units of agricultural products through a year of work in the fields. Although it is certain that actually with difference in one's ability, the same working hours throughout the year will result in a slight variance in one's harvest, here we will not refer to an individual difference like this. Rather what is noticeable in the pre-modern society is variance in annual harvests. In the pre-modern society, because of bad weather or the outbreak of plant diseases, annual harvests, regardless of the same amount of labor input, greatly fluctuate from year to year. Consequently, even if this year's crop is normal, we are little assured that we shall have a normal crop and obtain the average agricultural products X^e next year as well; agricultural production is quite uncertain. A crop yield X every year is, at the moment of planting, a random variable which will be significantly influenced by natural conditions. Whether one's toil and trouble in the fields will be repaid depends on natural conditions including weather. Nevertheless, with the same natural conditions, a crop yield will be in proportion to one's working hours; under the same natural conditions, if you cut your working hours by half, a crop yield is also reduced by half. Meanwhile, in industrial production also, after taking a job training for several hours or more, anyone can produce goods; there is no difference in that between agricultural and industrial production. But, the degree of dependence on natural conditions in industrial production is smaller than in agricultural production, and the variance of output in industrial production is very small. Therefore, the output of industrial production, in contrast to that of agricultural production, is not a random variable.

Since anyone, as we have already stated, potentially has some abilities to pro-

duce either agricultural or industrial products, what role you will play in the social production depends on how you allocate your working hours between two distinct sectors of the social production. Now let's standardize each annual working hours to unity[5]. Those who allocate 1 hour of the whole working hours to agricultural production are peasants; that is to say, using $\bar{D}$ units of industrial products including farming tools[6] each peasant reaps annually X units of agricultural products through 1 hour of work in the fields. At the same time, those who allocate part of 1 hour of the whole working hours to industrial production are craftsmen; that is to say, each craftsman divides 1 hour of the annual working hours into u hours of farm working and $1-u$ hours of handiwork, where we assume that $0 < u < 1$. Although it is certain that the division of the annual working hours into agricultural and industrial labor considerably varies from one type of occupation to another in handicrafts, here we will leave out of consideration such differences in the division of the annual working hours. Since, under the same natural conditions, a crop yield is in proportion to labor input in agricultural production, a crop yield uX is obtained by u hours of agricultural labor, and industrial products $u\bar{D}$ is spent in the process of farm working. In conclusion, each craftsman, making use of industrial products $u\bar{D}$, reaps annually agricultural products uX through u hours of farm working as well as producing industrial products Y through $1-u$ hours of handiwork. While peasants devote themselves solely to cultivation and stock raising, craftsmen work hard not only in their workshops but also in the fields; even a craftsman is not allowed to be independent of cultivation and stock raising. Thus the division of labor between agriculture and industry in the pre-modern society is far from complete.

Both peasants and craftsmen, the former producing agricultural products and

the latter producing agricultural and industrial products, contribute to the social production. Then, how about the expenditure of agricultural and industrial products produced by peasants and craftsmen? Because, as is well known, three economic agents of the seignior, the peasant, and the craftsman constitute the economic structure of the pre-modern society in which "the rule over the land" is noticed, we must refer to the consumption and the investment of all the three economic agents in order to depict the whole use of products in the pre-modern society. But, leaving the consideration of the behavior of the seignior to the sections below[7], here, first of all, we will deal with that of peasants and craftsmen.

Whether he is a peasant or a craftsman, everyone needs at least a fixed amount of annual food consumption $\overline{C}$ for his existence. Through a year of agricultural labor the peasant produces every year farm products X, which exceed the minimum food consumption required for existence $\overline{C}$; i.e. we have

$$X > \overline{C} .$$

In reality, it may happen that even a peasant can not provide himself with necessary foodstuffs in a year of a severe crop failure; yet we will not suppose such an extreme case in this paper. On the other hand, the craftsman can hardly ever provide himself with necessary foodstuffs. It is sure that the craftsman also works in the fields for u hours a year, but farm products uX produced by u hours of agriculture labor fall short of the required food consumption $\overline{C}$, i.e.

$$uX < \overline{C} .$$

Although, of course, a craftsman might be able to provide himself with foodstuffs necessary for existence in a year of an extremely bountiful harvest, actually this will be a fairly exceptional situation.

A few industrial products are no less indispensable for productive activities in

the pre-modern society than farm products are for one's existence. Agricultural production in the pre-modern society is characterized by the use of simple farming tools and wide applications of livestock. With such agricultural technology given, it is inevitable that industrial products including farming tools are used in a productive process. In fact, in our hypothetical situation also, the peasant spends industrial products $\overline{D}$ in 1 hour of farm working, and the craftsman $u\overline{D}$ in u hours of farm working. Thus, in order to replace industrial products used up or to further extend the existing scale of production, freshly produced industrial products are put into the process of agricultural production. Needless to say, the use of industrial products is not restricted to productive processes; a large number of industrial products enter one's daily consumption in the form of processed foods, clothes, furniture, accessories, etc.

Anyway, industrial production is the productive process in which, using mineral resources and agricultural products obtained from nature as raw materials, we work them further. To be loyal to this definition, we can not help but take into account the input of raw materials in the case of industrial production as well as in that of agricultural production. Nevertheless, in this paper we would not clearly mention the input of raw materials, agricultural products included, into industrial production. The explicit presentation of the input of raw materials into the process of industrial production might by itself make the equation between supply and demand of farm products introduced below, more complicated. Yet, on the other hand, this simplification will correct no substantial part of our conclusion. We, more than anything else, avoid the argument being made unnecessarily complicated.

3. The steady state

The peasant taking charge of agricultural production and the craftsman taking charge of industrial production, the social division of labor is made between agriculture and handicraft in the pre-modern society. Then, are the peasant and the craftsman, both in charge of productive activities, able to maintain by themselves this system of the social division of labor between agriculture and industry? If the system can be maintained by the sheer force of peasants and craftsmen, there will be no room any longer for the third party to intervene in the system; but can that really be the case?

First of all, we will make clear what is meant by the continuance of the system of the social division of labor between agriculture and industry. In the widest sense, the continuance of the system means that the social division of labor is maintained between agriculture and handicraft. Yet, we will understand the word in the narrowest sense; namely the continuance of the system is the repetition of exactly the same scale of economic activity in the system of the social division of labor between agriculture and industry. In so far as economic activity repeats itself exactly in the same scale, the system of the social division of labor once established never breaks down. As a matter of facts, since a crop yield widely fluctuates from year to year in highly volatile agricultural production in the pre-modern society, there is no guarantee that the same scale of economic activity will repeat itself every year; therefore, to put it exactly, the continuance of the system of the social division of labor means that, over a sufficient number of years, exactly the same scale of economic activity is repeatedly restored on average. If the situation in which exactly the same scale of economic activity repeats itself on average is called the steady state, the continuance of the system of the social division of labor

is in other words the attainment of the steady state in the economy.

The following two conditions are met in the steady state. First, the size and the composition of the population remain constant in the economy; namely the population of both peasants and craftsmen are invariable. Second, the amounts of agricultural and industrial products remain constant; on average, each peasant reaps the annual harvest X^e, and each craftsman produces agricultural products uX^e and industrial products Y yearly.

As we have already stated, no one can survive without a fixed amount of food $\overline{C}$; accordingly, in order to maintain at least a given size of population, the community has to secure sufficient food to support this population every year. Now, suppose that the community consists of m peasants and n craftsmen. Then, to maintain the whole population, the community needs their foodstuffs $m\overline{C} + n\overline{C}$ every year. If we could not carry over the harvest in each year to the next year, we would have to obtain the foodstuffs required every year from a crop yield in the same year. But, if it is possible to carry over the harvest in each year to the next year, we can slightly relax the requirement. For the sake of simplicity, assume that we can everlastingly store agricultural products without further costs[8]. Keeping sufficient food in reserve from the normal year to provide against emergencies, the community can relieve food shortages in lean years by breaking into the emergency food supplies. With sufficient food reserves, we no longer need to care if a crop yield in each year $mX + nuX$ exceeds the community's food requirements $m\overline{C} + n\overline{C}$ in the same year. If we can merely get an average of crop yields over sufficient years $mX^e + nuX^e$ greater than the community's food requirements $m\overline{C} + n\overline{C}$, that will be enough:

$$mX^e + nuX^e \quad m\overline{C} + n\overline{C} .$$

Anyway, if an expectation of annual harvests $mX^e + nuX^e$ exceeds the minimum requirements for existence $m\overline{C} + n\overline{C}$ in the whole society in the long run, the public will be better off to some extent. In fact, one may expend the agricultural surpluses either in expanding agricultural production or in improving one's quality of life.

With the agricultural surpluses in the current year, a further expansion of planting from the next year on will raise agricultural production; and a steady increase in food consumption also may lead to population growth in the end. In either case, with a variation in the scale of production or the size of population, the steady state will fail. Solely in the situation in which neither food surpluses nor food shortages exist, that is to say, only in the situation in which the annual harvest is on average equal to the minimum requirements for existence in the society as a whole, is the economy kept in the steady state. In other words, the realization of the steady state requires that the expectation of annual crop yields $mX^e + nuX^e$ be equal to the community's food requirements $m\overline{C} + n\overline{C}$:

$$mX^e + nuX^e = m\overline{C} + n\overline{C} \ . \tag{1}$$

We have already stated that industrial products including farming tools are spent in the process of agricultural production, and industrial products thus used up are replaced with new ones yearly produced. In fact, every year peasants expend industrial products $m\overline{D}$ in total in the process of agricultural production, and craftsmen also expend industrial products $nu\overline{D}$ in total. To maintain agricultural production we must replace the industrial products $m\overline{D} + nu\overline{D}$ in total used up in this process with the new products nY produced in the same year. Therefore, in the steady state we have an inequality,

$$nY \quad m\overline{D} + nu\overline{D} \ ,$$

regarding the industrial products yearly produced. Actually, if the industrial products nY produced every year are greater than the replacement demand $m\overline{D} + nu\overline{D}$ in agricultural production, one may consume the excess in one's everyday life or may invest it in the agricultural sector[9]. The excess of industrial products, however, being continuously invested in the agricultural sector, the average level of agricultural output will rise, and the economy will no longer stay in the steady state. Consequently, the steady state in the economy requires that the excess of the supply over the replacement demand of industrial products in the agricultural sector be totally consumed.

4. Free exchanges of products

In the preceding section we have defined the steady state and set out the conditions which the supply and the demand of both agricultural and industrial products should satisfy in the steady state. Is it possible to achieve the steady state by their efforts? With no one other than peasants and craftsmen, the resource allocation under the system of the social division of labor between agriculture and industry is determined solely by their free action. In this section, first of all, we will consider what economic states are brought about by free action of peasants and craftsmen.

Without the third party other than them, both peasants and craftsmen can freely seek their gains. Yet, they are not completely free from constraints around them. Both peasants and craftsmen as human beings can never neglect biological requirements for existence and can never escape technological conditions of production which, making the social division of labor, give each of them a definite economic role; In short, they act freely within the bounds of constraints comprising

biological requirements and technological conditions of production presented in section 2. Besides, some other constraints may also restrict one's behavior; in fact, one's behavior is restricted by the amount of information which one can obtain. In the pre-modern society the public are not furnished with sufficient information to foresee the remote future, and their time horizons are comparatively narrow. One takes into account solely the comparatively near future to take a decision. More specifically, we assume that every year either peasants or craftsmen draw up consumption schedules for the current year only and arrange how to deal with their agricultural and industrial products. In taking a decision, they do not take into consideration any circumstances from the next year on.

Within given constraints one may make the choice which best fits one's own preference. One's preference is represented by an utility function. To begin with, let's compose a utility function for the peasant. We have already stated that the peasant needs food consumption $\bar{C}$ for his existence and also has to replace the industrial products used up in the production process for the continuance of production. In the composition of the utility function we must take these points into account. Only in the case where the consumption of agricultural products C exceeds the consumption requirements $\bar{C}$, and that of industrial products D exceeds the replacement demand $\bar{D}$ in agricultural production, does the peasant have a possibility of choosing consumption goods. But, without securing the minimum requirements for existence $\bar{C}$ or without replacing completely the industrial products used up $\bar{D}$, the peasant finds no room for consumer's choice. In this case, he can not support his own material life, and, without sustaining life, whatever consumer's choice he makes is equally of no value to him. We will formulate the preference of the peasant above described in the form of the following

utility function $U_1 : R_+^2 \to R_+$:

$$U_1(C,D) = \begin{cases} H(C-\bar{C},\ D-\bar{D}) & (C > \bar{C} \quad \text{and} \quad D > \bar{D}) \\ 0 & (C \cdot \bar{C} \quad \text{or} \quad D \quad \bar{D}) \end{cases},$$

where the function $H : R_+^2 \to R_+$ is a utility function which represents consumer's preference order and satisfies the usual properties of the utility function. In addition, for $^\forall (x,y) \in R_{++}^2$ we have

$$H(x,y) > 0 ,$$

and furthermore the function U_1 is assumed to be continuous. We have already stated that the peasant has narrow time horizons and cares nothing for the standard of living from the next year on. It is a reflection of this point that the utility level U_1 is a function of the current consumption only and does not depend on consumption for the following periods.

When the social division of labor between agriculture and industry is incomplete, the craftsman, along with the peasant, works in the fields. Agricultural working hours, however, of the craftsman are shorter than those of the peasant, and a smaller amount of industrial products are used up in the process of farm work; the craftsman can continue with agricultural production by replacing barely industrial products $u\bar{D}$ every year. Except for this point, the preference of the craftsman is hardly different from that of the peasant. That is to say, only in the case where the consumption of agricultural products C exceeds the consumption requirements $\bar{C}$ and the demand for industrial products D exceeds the replacement demand $u\bar{D}$ in agricultural production, is there some possibility that the craftsman chooses consumption goods. The preference of the craftsman such as this will be formulated by means of the following utility function $U_2 : R_+^2 \to R_+$:

$$U_2(C,D) = \begin{cases} H(C - \bar{C}, \ D - u\bar{D}) & (C > \bar{C} \quad \text{and} \quad D > u\bar{D}) \\ 0 & (C \cdot \bar{C} \quad \text{or} \quad D \quad u\bar{D}). \end{cases}$$

Every year the peasant reaps agricultural products X, which the peasant can deal with completely at his discretion; either the peasant himself can consume all the harvests or he can dispose of the agricultural surpluses exceeding the requirements for his subsistence $\bar{C}$ at price p in order to purchase industrial products D. Of course, at the moment of consumer's choice, the peasant obeys the budget constraint,

$$pX = pC + D,$$

where the price of industrial products is set at unity[10]. At a given price of farm products p [11], the peasant determines the consumption of farm products C and industrial products D so as to maximize the utility level U_1 under the budget constraint, which, mathematically, amounts to solving the following optimization problem:

$$\text{Max } U_1(C,D)$$
$$\text{s.t. } pX = pC + D.$$

Meanwhile, the income of the craftsman consists of farm products uX and industrial products Y, and his expenditure comprises the demand for food C and that for industrial products D; the budget constraint of the craftsman can be written as

$$puX + Y = pC + D.$$

With the price of agricultural products p given, the craftsman determines the consumption of farm products C and industrial products D so as to maximize the utility level U_2 under the budget constraint; that is to say, the craftsman solves the optimization problem:

$$\text{Max} \quad U_2(C,D)$$
$$\text{s.t.} \quad puX + Y = pC + D.$$

In this manner, at given price of agricultural products p the peasant and the craftsman each solve their optimization problems to settle the demands for agricultural and industrial products. Yet, it is not the case that, at any price of agricultural products p, the free exchange of goods sets off between peasants and craftsmen; without an expectation that the level of utility will rise, the peasant as well as the craftsman would not alter the composition of their own property through taking part in the exchange of products. The peasant owns only farm products X at an initial point of time. At this moment, according to his utility function, the peasant can raise his utility level only in the situation where he attains an alteration in the composition of his property to have food consumption C greater than the minimum requirements $\bar{C}$ as well as industrial products D greater than the replacements $\bar{D}$ in agricultural production. (See Figure 1) In view of his budget constraint, for the peasant to be able to alter his property holdings in this direction in consequence of the exchange of goods the price of agricultural products p must satisfy

$$pX > p\bar{C} + \bar{D}. \tag{2}$$

(Figure 1 is around here)

Correspondingly, the craftsman owns farm products uX and industrial products Y at an initial point of time. Then the craftsman can raise his utility level only in the situation where his food consumption C exceeds the minimum requirements $\bar{C}$ and the demand for industrial products D becomes greater than the replacements $u\bar{D}$ in agricultural production. (See Figure 2) Taking the craftsman's budget constraint into account, we find that the attainment of such composition of

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products requires that the price of agricultural products p satisfy

$$puX + Y > p\bar{C} + u\bar{D}. \qquad [3]$$

(Figure 2 is around here)

From the expression [2] and [3], we have a range of the agricultural price p:

$$\frac{\bar{D}}{X - \bar{C}} < p < \frac{Y - u\bar{D}}{\bar{C} - uX}.$$

Unless the agricultural price p falls in this range, a voluntary exchange of agricultural and industrial products will never take place between peasants and craftsmen.

Once an exchange of agricultural and industrial products is set in motion, a market mechanism in each market functions to achieve an equilibrium between supply and demand. To begin with, we will describe the state of equilibrium in the agricultural market, where, farm products mX being produced by m peasants and farm products nuX being produced by n craftsmen, farm products $mX + nuX$ in total are supplied to the whole society. On the other hand, the peasant determines an agricultural demand $C_1(p, X)$ so as to maximize his utility level at a given price p and given farm products X, and the craftsman determines an agricultural demand $C_2(p, X, Y)$ so as to maximize his utility level with a price level p, farm products uX, and industrial products Y given. Their demands for farm products add up to $mC_1(p, X) + nC_2(p, X, Y)$ in the whole society, and we have

$$mX + nuX = mC_1(p, X) + nC_2(p, X, Y) \qquad [4]$$

in equilibrium in the agricultural market. Then, we will move on to the industrial market. The total supply in this market consists of merely industrial products nY produced by n craftsmen. At the same time, the peasant determines an industrial

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demand $D_1(p,X)$ so as to maximize his utility level at a given price p and given farm products X, and the craftsman determines an industrial demand $D_2(p,X,Y)$ so as to maximize his utility with a price level p, farm products uX, and industrial products Y given. The total demand for industrial products in the society amounts to $mD_1(p,X)+nD_2(p,X,Y)$, and the equality between supply and demand in the industrial market turns out to be

$$nY = mD_1(p,X)+nD_2(p,X,Y) .\qquad [5]$$

In addition, it goes without saying that if one of the two equalities between supply and demand holds, the other must hold by Walras' law.

   Solving either equation [4] or [5], we get the equilibrium price of farm products p^* and a new allocation of resources consequent on the exchange of goods. The process of the exchange can be illustrated with Edgeworth box diagram; in Figure 3, setting m peasants' origin at the south-west corner O_1 of the diagram and n craftsmen's origin at the north-east corner O_2, we draw their budget constraints and indifference curves. As a result of the exchange between agricultural and industrial products, the allocations of their property shifts from the initial point A to the equilibrium point B. As we find immediately from the diagram, each peasant provides craftsmen with the agricultural surpluses $X-C_1$ to get industrial products D_1 in this transaction. In fact, the agricultural demand C_1 and the industrial demand D_1 submit to the budget constraint

$$pX = pC_1 + D_1.$$

(Figure 3 is around here)

At the moment, noting one of the conditions under which the exchange of goods commences,

$$D_1 > \bar{D},$$

we get

$$X > C_1,$$

from which we find that the peasant certainly supplies himself with his food consumption C_1 from his own crop yield X and still can furnish craftsmen with the agricultural surpluses $X - C_1$. On the other hand, each craftsman obtains food deficiencies $C_2 - uX$ from peasants in exchange for the surpluses of industrial products $Y - D_2$. In fact, the agricultural demand C_2 and the industrial demand D_2 must satisfy the craftsman's budget constraint

$$puX + Y = pC_2 + D_2.$$

Here, since, taking account for one of the conditions under which the exchange of goods starts, i.e.

$$C_2 > \bar{C}$$

and the fact that the harvests reaped by the craftsman uX fall short of the minimum requirements for existence $\bar{C}$, i.e.

$$uX < \bar{C},$$

we get

$$Y > D_2,$$

it is confirmed that, certainly securing a portion for his own consumption D_2 out of a stock of industrial products Y, the craftsman still can provide peasants with the surpluses of industrial products $Y - D_2$.

5. The sustenance of the system of the social division of labor

In the preceding section we have considered what allocation of goods takes place every year as an outcome of free exchanges of products between peasants and craftsmen. Then, will the system of the social division of labor between agriculture

and industry be sustained through such free exchanges? We have already taken the sustenance of the system of the social division of labor in its stricter sense as a realization of the steady state. Will the allocation of goods resulting from free exchanges of products be compatible with the conditions for the steady state?

As we have already stated, annual yields of the harvest are by no means stable in the pre-modern society. Taking this point into account, we have made the amount of agricultural production X a random variable; the harvests depend on natural conditions, and we can not foresee the amount of production X in advance. Nevertheless, even the random variable X has an upper and a lower bound. While the amount of agricultural production X exceeds the minimum requirements for existence $\overline{C}$, the farm products uX reaped by the craftsman do not attain the minimum requirements $\overline{C}$; that is to say, the amount of agricultural products X satisfies the inequality

$$\overline{C} < X < \frac{1}{u}\overline{C} \ .$$

Within this range the amount of agricultural products X has a continuous probability distribution which, for example, is shown in Figure 4. The probability density function of this distribution is given by a function $f : R_+ \rightarrow R_+$.

(Figure 4 is around here)

With farm products X and industrial products Y given, the free exchanges of products between peasants and craftsmen have established an equilibrium between supply and demand in the agricultural market each year, and the equilibrium price of farm products p^* is determined. As seen immediately from the equation[4], with industrial products Y given, the equilibrium price of farm products p^* thus determined is a function of farm products X, and, in addition,

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both food consumption of the peasant C_1 and that of the craftsman C_2, which correspond to the equilibrium price p^*, come to be a function of farm products X; both food consumption of the peasant and that of the craftsman depend on the crop yield X of the same year and therefore become random variables, which will have different values according to an increase or a decrease in the crop yield.

The peasant and the craftsman determine their food consumption depending on volatile agricultural production. Then, can the steady state of the society be maintained by such free choices of consumption goods?

For the time being, taking no account whether these are attained as an outcome of free choices by the public, let's consider what values the peasant's consumption C_1 and the craftsman's consumption C_2 will have when the economy remains at the steady state. Needless to say, since the existence of the peasant and the craftsman is guaranteed,

$$\mathrm{C}_1(p,X) \geq \bar{C}, \qquad \mathrm{C}_2(p,X,Y) \geq \bar{C} \qquad\qquad [6]$$

hold good in the steady state. In actual fact, however, the amounts of each con-sumption which are compatible with the steady state are found to be nothing but the minimum requirements $\bar{C}$. To demonstrate this proposition, suppose that the opposite is true; namely suppose that for a level of agricultural production X,

$$\mathrm{C}_1(p,X) > \bar{C}, \qquad \mathrm{C}_2(p,X,Y) > \bar{C}. \qquad\qquad [6a]$$

As we have shown in the preceding section, the food consumption of the peasant C_1 and that of the craftsman C_2 satisfy

$$mX + nuX = mC_1(p,X) + nC_2(p,X,Y)$$

every year. Here, noting that both the food consumption of the peasant C_1 and that of the craftsman C_2 are random variables, take the expectation of both sides of the equality to get

$$mX^e + nuX^e = mE\big[C_1(p,X)\big] + nE\big[C_2(p,X,Y)\big].$$

In view of the supposition [6a], we immediately get[12]

$$mX^e + nuX^e > m\overline{C} + n\overline{C} ,$$

which contradicts the necessary condition for the steady state shown in the section 3:

$$mX^e + nuX^e = m\overline{C} + n\overline{C} . \qquad [1]$$

Therefore, we find that our proposition [6a] is wrong and that correctly for any level of farm products X we have

$$C_1(p,X) \cdot \overline{C}, \qquad C_2(p,X,Y) \quad \overline{C} .$$

Finally, noting [6], it is easy to get

$$C_1(p,X) = \overline{C}, \qquad C_2(p,X,Y) = \overline{C} .$$

In the last analysis, the assumption of the steady state implies that the food consumption of the peasant and that of the craftsman correspond to the minimum requirements for existence $\overline{C}$ irrespective of whether there are more or less crop yields.

Conversely, if food consumption of the public is always equal to the minimum requirements $\overline{C}$, the steady state will be brought about to maintain the system of the social division of labor between agriculture and industry.

Provided that the amount of food consumption of the public remains at the subsistence level $\overline{C}$ even in years of abundance, the community can carry the farm products which are not consumed these years over to the subsequent years. In years of abundance, the amount of agricultural production X exceeds the expectation of production X^e in the fields, and in the society as a whole the total agricultural products $mX + nuX$, which are greater than the expectation of the total yields $mX^e + nuX^e$, will be harvested. From [1], the expectation of the total

yields amounts to the sum of the minimum requirements for existence. Therefore, in years of abundance, the total yields $mX + nuX$ become greater than the sum of the minimum requirements $m\overline{C} + n\overline{C}$, and the community can keep in reserve the difference between the realized value $mX + nuX$ and the expected value $mX^e + nuX^e$ of the total yields:

$$(m + nu)(X - X^e).$$

On the other hand, in a lean year, the amount of agricultural production X gets smaller than the expectation of production X^e, and the total yields of farm products $mX + nuX$ fall short of the minimum necessary food consumption for the whole society $m\overline{C} + n\overline{C}$. In the community, the food shortages amount to

$$(m + nu)(X^e - X).$$

Nevertheless, sufficient food reserves kept from normal years can solve this food problem, because, keeping sufficient farm products in reserve from normal years, the community can make up the deficiencies in foodstuffs in lean years out of farm products in reserve.

In fact, satisfying

$$0 \quad Y^+ < \overline{Y}^+ = (m + nu)\left(\frac{1}{u}\overline{C} - X^e\right),$$

the food reserves Y^+ have an absolutely continuous distribution whose probability density function is

$$g(y^+) = f\left(X^e + \frac{y^+}{m + nu}\right) \cdot \frac{1}{m + nu}.$$

On the other hand, satisfying

$$0 < Y^- \leqslant \overline{Y}^- = (m + nu)\left(X^e - \overline{C}\right),$$

the food shortages Y^- have an absolute continuous distribution whose probability

density function is given by

$$g(y^-) = f\left(X^e - \frac{y^-}{m+nu}\right) \cdot \frac{1}{m+nu} .$$

Let's calculate the difference between the expectation $E(Y^+)$ of the food reserves Y^+ and the expectation $E(Y^-)$ of the food shortages Y^-:

$$
\begin{aligned}
&E(Y^+) - E(Y^-) \\
&= \int_0^{\bar{Y}^+} y^+ g(y^+) dy^+ - \int_0^{\bar{Y}^-} y^- g(y^-) dy^- \\
&= \int_{X^e}^{\frac{1}{u}\bar{C}} (m+nu)(x - x^e) f(x) dx - \int_{X^e}^{\bar{C}} -(m+nu)(x^e - x) f(x) dx \\
&= (m+nu)\left[\int_{\bar{C}}^{\frac{1}{u}\bar{C}} (x - x^e) f(x) dx \right] \\
&= 0
\end{aligned}
$$

Therefore, the expectation of the food reserves is equal to that of the food shortages:

$$E(Y^+) = E(Y^-) .$$

This means that sufficient food reserves from normal years prevent famines from occurring in lean years. Then, who is actively engaged in increasing a reserve of foodstuffs from normal years, and who receives a benefit from it?

Since, concerning the amount of farm products X, we have supposed

$$\bar{C} < X < \frac{1}{u}\bar{C} ,$$

the peasant will not suffer from famines even in lean years while the craftsman has food shortages. In what way does the craftsman solve his food problem? Rewriting the necessary condition in the steady state [1] as

$$m(X^e - \bar{C}) = n(\bar{C} - uX^e) ,$$

we find that, in general, craftsmen's deficiencies in food $n(\bar{C} - uX^e)$ are supplied

by peasants' agricultural surpluses $m(X^e - \overline{C})$. In lean years, however, the total crop yields will be lower than the sum of the minimum necessary food consumption:

$$mX + nuX < m\overline{C} + n\overline{C} \ .$$

Rearranging this to get

$$m(X - \overline{C}) < n(\overline{C} - uX) \ .$$

Thus, in a lean year craftsmen's deficiencies in food $n(\overline{C} - uX)$ are not made up for only by peasants' agricultural surpluses $m(X - \overline{C})$ in the same year. If that is the case, a portion of food deficiencies must be supplied by the food reserves from normal years. Since, according to our assumption, the peasant can supply himself with the minimum necessary food even in lean years, he has no need whatever to keep a reserve of food for himself. The reserve of food which the peasant has kept from normal years will be exclusively supplied to craftsmen in lean years; from normal times the peasant is actively engaged in increasing food reserves for the sake of craftsmen, who will suffer from food shortages in crop failure.

Certainly, if food consumption of the public is always limited to the minimum requirements for existence to have agricultural surpluses in reserve, the economy will attain the steady state. But, is the public willing to accept his everyday consumption which barely stays at the subsistence level? Moreover, is the peasant's commitment to food reserves not for improvement in his own living standard but for the sustenance of craftsmen's existence truly in accordance with his profits?

With the agricultural price p in the fixed range, the peasant, who makes a free choice of consumption goods, voluntarily takes part in the exchanges of products to alter his property holdings. According to the utility function presented in the

preceding section, any pair of products (C_1, D_1) thus reached has higher utility than the pair of products $(\overline{C}, D_1)$, where food consumption remains at the subsistence level, to say nothing of the resource allocation before exchanges $(X, 0)$:

$$U_1(C_1, D_1) > U_1(X, 0) = U_1(\overline{C}, D_1).$$

Therefore, to restrict food consumption to the minimum necessary level $\overline{C}$ with a further alteration to the property holdings (C_1, D_1), which result from the exchanges of products, is to lower the utility level of the peasant and is clearly contrary to his profits. Since beyond doubt there is no room whatever for a free peasant to accept an alternative contrary to self-interest, we can not restrict the peasant's food consumption to the minimum requirements for existence without suppressing his free will. The sustenance of the steady state compels the peasant to lower his standard of living. As for the craftsman, the circumstances are the same. We can not help but suppress the free will of the craftsman in order to restrict his food consumption to the minimum necessary level and thus maintain the steady state.

To sum up, in order to sustain the steady state and maintain the system of the social division of labor between agriculture and industry, we must, suppressing the free wills of the peasant and the craftsman, control the free exchanges between agricultural and industrial products. Those who will suppress the free will of the peasant are economic agents clearly different from peasants, and those who will compel the craftsman to accept his consumption which he does not wish are economic agents clearly different from craftsmen; that is to say, an economic agent who will intervene in the exchanges between agricultural and industrial products is the third party other than either the peasant or the craftsman.

Let's call the third party, who exerts a compelling power over peasants and

craftsmen to force them to alter the resource allocation consequent on free exchanges of products, the seignior. As for farm products, the seignior restricts the food consumption of the peasant and the craftsman to the minimum necessary level against their wills as well as levying agricultural surpluses on peasants to keep them in reserve from normal times and distributing the foodstuffs in reserve to craftsmen in lean years[13]. On the other hand, as for industrial products, the seignior takes care that the industrial products made by craftsmen are well supplied to peasants and that farming tools used up in the process of agricultural production are adequately replaced. Intervening in the direct exchange between agricultural and industrial products against the free wills of the peasant and the craftsman, the seignior applies himself to uninterrupted reciprocal supplies of products between them.

In a word, the economic function of the seignior consists in intervening in the exchange between agricultural and industrial products under circumstances of volatile agricultural production; this economic function is indispensable for the sustenance of the system of the social division of labor between agriculture and industry. In addition, since his economic function is indispensable, the presence of the seignior as the agent for this function, on the premise that the system of the social division of labor between agriculture and industry is maintained, is no less inevitable than that of the peasant and the craftsman is.

6. The economic structure of the seigniorial system
The system of the social division of labor between agriculture and industry can not be maintained solely with the help of participants in productive activities such as peasants and craftsmen; the existence of the third economic agent, the seignior, is

indispensable for the sustenance of the system of the social division of labor. Then, when the seignior, along with the peasant and the craftsman, occupies a firm position in the system to promote the exchange between agricultural and industrial products, what will become of the resource allocation in the society? In this section, we will explore how the social products, in particular farm products of the society, will be allotted to seigniors, peasants, and craftsmen.

Suppose that ℓ seigniors exist in the society. Since the seignior is not engaged in agricultural production, his presence will not imply the increase in the total supply of farm products in the society. At the same time, with his presence, the total demand of farm products increases by his food consumption. Thus, nothing other than an increase in the population of peasants, the sole economic agents who are able to provide agricultural surpluses to others, can establish the equality between the supply and the demand of farm products. At this moment, let the population of peasants be m'. Then, the total supply of farm products becomes $m'X + nuX$; moreover, on the supposition that the seignior eats food v times ($v > 1$) as much as the minimum requirements level $\overline{C}$ on average[14], the aggregate demand of farm products amounts to $m'\overline{C} + n\overline{C} + \ell v\overline{C}$. In the steady state the expectation of the total supply of farm products $m'X^e + nuX^e$ is equal to the total demand $m'\overline{C} + n\overline{C} + \ell v\overline{C}$:

$$m'X^e + nuX^e = m'\overline{C} + n\overline{C} + \ell v\overline{C} . \qquad \text{[1a]}$$

Comparing this expression with the necessary condition of the steady state,

$$mX^e + nuX^e = m\overline{C} + n\overline{C} , \qquad \text{[1]}$$

which we have given in section 3, we immediately get

$$(m' - m)(X^e - \overline{C}) = \ell v\overline{C} .$$

Here, noting that $X^e > \overline{C}$, we find that the presence of the seignior, i.e.

$$\ell > 0$$

implies

$$m' > m \ .$$

With the presence of the seignior, the peasant population has to be greater than otherwise. Because of added food consumption by seigniors, agricultural surpluses, which all the peasants should provide for the public, increase; more peasants will work vigorously in the fields than before, and larger agricultural surpluses will be supplied for the public.

As we have confirmed in the preceding section, if the community, which consists of only peasants and craftsmen, could keep in reserve farm products in excess of the minimum requirements for its existence from normal times, it could provide for food shortages at times of poor harvests. When seigniors, along with peasants and craftsmen, take part in the community, is it also possible for the community to provide for food shortages at times of crop failures? In years of abundance, the community reaps farm products $m'X + nuX$, which exceed the minimum requirements for its existence $m'\overline{C} + n\overline{C} + \ell v\overline{C}$. At this moment, agricultural surpluses in excess of the minimum requirements in the society amount to

$$m'X + nuX - (m'\overline{C} + n\overline{C} + \ell v\overline{C}),$$

which, in view of [1a], can be further rewritten as

$$(m' + nu)(X - X^e) \ .$$

On the other hand, in lean years, the total yields of farm products $m'X + nuX$ fall short of the sum of the minimum requirements for existence $m'\overline{C} + n\overline{C} + \ell v\overline{C}$; the community suffers food deficiencies

$$(m' + nu)(X^e - X) \ .$$

Nevertheless, if we have always kept in reserve agricultural surpluses, we can

solve food shortages in times of poor harvests. In fact, from a calculation similar to that in the preceding section it follows that the expectation of agricultural surpluses is equal to that of food deficiencies:

$$E\left[(m' + nu)(X - X^{e})\right] = E\left[(m' + nu)(X^{e} - X)\right].$$

Consequently, keeping in reserve a portion of farm products which he have levied on peasants, the seignior can provide for food shortages in an emergency. Yet, what is feasible does not necessarily means that this actually takes place. First, for the sake of simplicity, we have made some rather unrealistic assumptions. As for food reserves, we have supposed that we can store up farm products everlastingly without additional costs. This, needless to say, never happens actually. For instance, we need storage facilities like granaries for the storage of grain and, however carefully they are kept, farm products will rot in the long run as well as be damaged by rats and harmful insects. Second, even if, with all our assumptions correct, we can keep in reserve a sufficient amount of farm products, there is no guarantee that the actual seignior plays his economic role adequately. The actual seignior may fail to evaluate properly the risk of a food crisis in the future, or may fail to suppress the peasant's and the craftsman's desire for consumption. Otherwise, solely for the purpose of improving his own standard of living, the seignior may use up the farm products he had collected from peasants. Anyway, whether or not a sufficient amount of farm products is kept in reserve depends on the will and the competence of the seignior in this case.

Besides, even in actual pre-modern societies, a few lords may not fulfill the economic role which is allotted to them. As we have confirmed, however, the system of the social division of labor between agriculture and industry can not survive without the economic function of the seignior, and, apart from technical impedi-

ments, the seignior can satisfactorily fulfill the economic function in sustaining the system of the social division of labor.

7. Concluding remarks

In this paper, constructing a mathematical model on the assumption that the system of the social division of labor between agriculture and industry is well established, we have demonstrated that the presence of the seignior is indispensable for the continuance of the system. When part of industrial production requires job skills of a high order, peasants taking charge of agricultural production and craftsmen taking charge of part of industrial production, the social division of labor is made between agriculture and industry. In addition, when the volatility of agricultural production is too high to disregard, the system of the social division of labor between agriculture and industry can not survive without the presence of the seignior.

Let's briefly look at again the process of demonstration. First of all, we have made it clear what the sustenance of the system of the social division of labor between agriculture and industry exactly means. The sustenance of the system means the preservation of the steady state. Further, after making this preparation, we have explored what allocation of resources comes about as a result of free action of both peasants and craftsmen. But, with highly volatile agricultural production, the resource allocation thus brought about will prevent the economy from staying at the steady state. That is to say, in so far as both the peasant and the craftsman are allowed to make a free choice of consumption goods, we can not sustain the system of the social division of labor between agriculture and industry. Therefore, for its continuance we must restrict the peasants' and the craftsmen's

choices of consumption goods, and the presence of the seignior, the third economic agent who restricts their action, is indispensable.

The decisive factor which, with a clear distinction between the peasant and the craftsman, causes the social division of labor between them is the need for job skills in part of industrial production. Presupposing this, we have demonstrated in this paper that the uncertainty in agricultural production is the factor which causes the seignior to perform the economic function of intervening in the exchange between agricultural and industrial products. The economic structure of the seigniorial system consisting of seigniors, peasants, and craftsmen is built on the particular type of production technology with definite properties. In the final analysis, what we have made clear is the correspondence between production technology and economic structure in the seigniorial system. Historically, production technology and economic structure in actual seigniorial systems have their diverse origins, and their formation can take its own course. But, whatever course their formation may take, once production technology and economic structure in the seigniorial system have been well established, there is no room for diversity in the correspondence between them.

Note:

1) Sekine[2000], Sekine[2003a], Sekine[2003b], Sekine[2003c].

2) Sekine[2003a], p.114. The exchange between agricultural and industrial products, however, does not necessarily mean a transaction in a market; still less the intervention in the exchange by the seignior is not restricted to commercial activities.

3) Sekine[2003a], pp.101-102, pp.142-146.

4) Sekine[2003a], pp.110-111, pp.147-153.

5) Although actually peasants are also engaged in industrial work requiring little job skills, for the sake of simplicity, we will not refer to this point in this paper.

6) Beyond doubt we have $\bar{D} > 0$.

7) The reason for this is because we can not logically derive the presence of the seignior at this point in time.

8) We will discuss the validity of this assumption below.

9) We have excluded the possibility that farm products are used as raw materials for industrial production.

10) Although we have explicitly refer to prices, we do not necessarily think that money functions as a medium of exchange between agricultural and industrial products, which we will state below.

11) The price of farm products p being given in the peasant's choice of consumption goods, each individual peasant can not change the price. In other words, we suppose that perfect competition prevails among exchanges between agricultural and industrial products. Nevertheless, this never implies that exchanges between agricultural and industrial products in an actual seigniorial system are carried out under perfect competition. Provided that both peasants and craftsmen act freely and that market mechanism works with the exchange between agricultural and industrial products, what resource allocation will take place? This is what we have wanted to know. For this reason, we have left out of consideration, to the greatest extent, circumstances that impede free exchanges of products or make the analysis unnecessarily complex.

12) Strictly speaking, this requires the condition that both the demand function of the peasant $C_1(p, X)$ and that of the craftsman $C_2(p, X, Y)$ are a continuous

function of farm products X. In microeconomics it is known that the consumer's demand function is continuous for prices and income under the usual utility function. (Varian[1984], p.163.)

13) The economic activities of lords, however, in actual pre-modern societies are never restricted to storages of agricultural surpluses.

14) This implies that as for farm products the consumption level of the seignior is higher than that of the peasant and the craftsman, who put up with the minimum requirements for existence. Actually, the income level of the seignior is substantially high in comparison with that of the peasant and the craftsman in many pre-modern societies. Then, why is the income level of the seignior higher than that of the peasant and the craftsman? Besides, what determines income differentials between the seignior and the peasant, or those between the former and the craftsman? Although these problems concerning income distribution in the pre-modern society are interesting in themselves, the problem of income distribution lies beyond the scope of our investigation in this paper.

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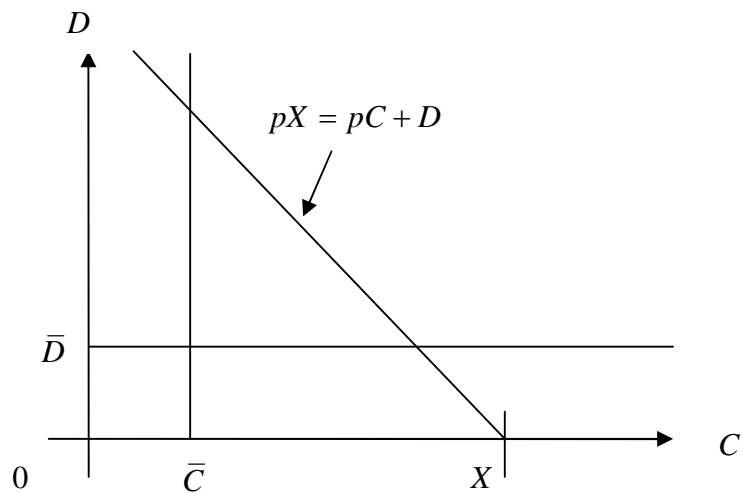Fig.1.   The budget constraint of the peasant

$$pX = pC + D$$

D (vertical axis), *C* (horizontal axis), $\bar{D}$, $\bar{C}$, X, 0

Fig.2.   The budget constraint of the craftsman

$$puX + Y = pC + D$$

D (vertical axis), *C* (horizontal axis), Y, $u\bar{D}$, uX, $\bar{C}$, 0
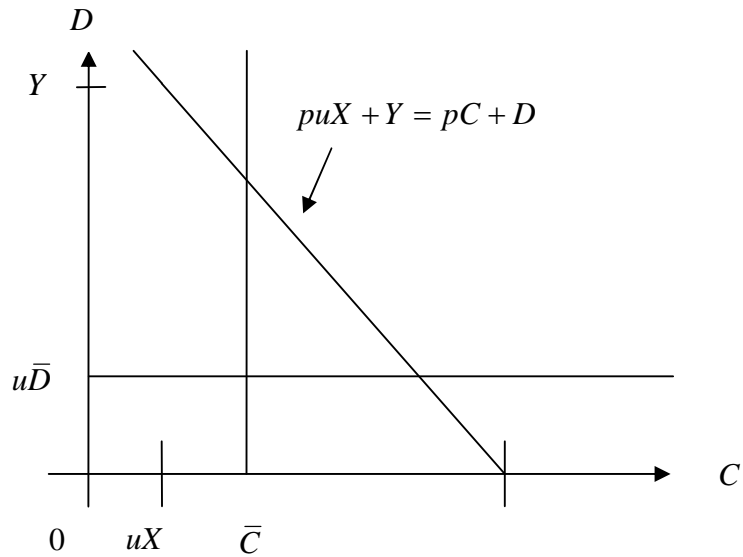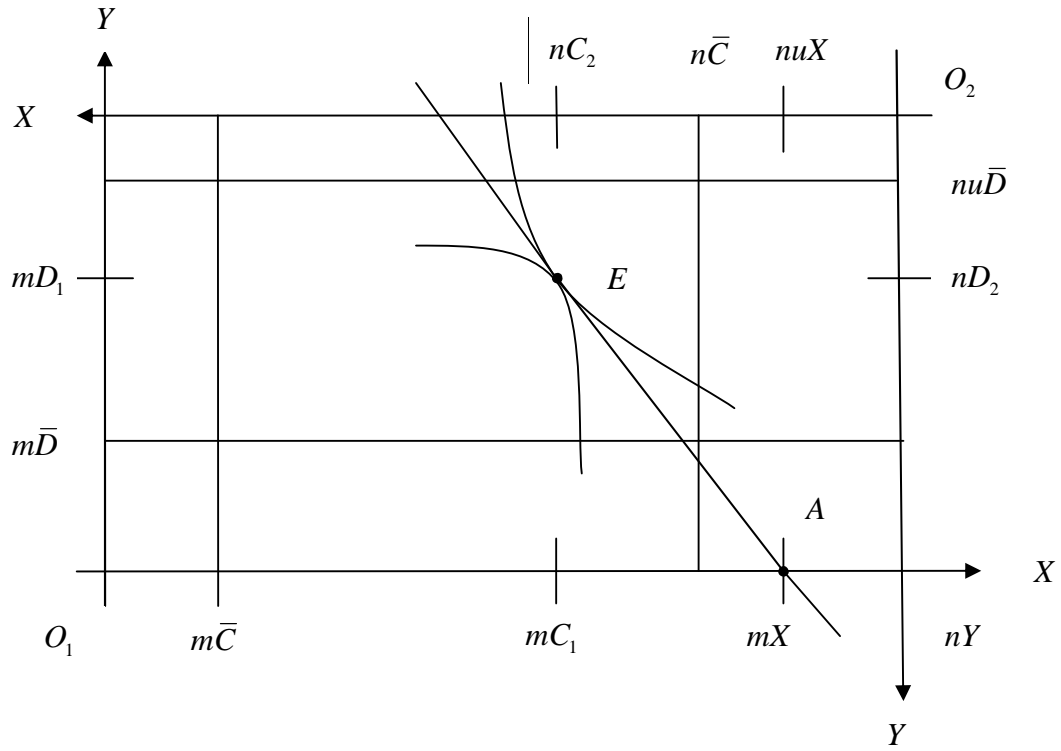
Fig.3. Box diagram

Fig.4. The probability density function with the farm products X