The Second Derivative Envelope Property and Theories of Demand

By Yoshihiko Otani*

Faculty of Economics

Kyushu Sangyo University

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[Abstract]

In this paper, we first obtain an interesting second derivative property of an optimum value function which we call the second derivative envelope property. We then show the usefulness of this property by applying it to the derivation of well-known properties of the Hicksian compensated demand function, the Slutskian compensated demand function and the indirect utility function in the standard demand theory. Finally we apply this approach in the investigation of properties of demand functions for goods and financial assets when agents face multiple budgets in incomplete financial asset markets.

Keyword: optimum value function, envelope theorem, demand theory, financial assets

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* Faculty of Economics, Kyushu Sangyo University, 2-3-1 Matsukadai, Higashi-ku, Fukuoka 813-8503, JAPAN. e-mail: otani@ip.kyusan-u.ac.jp

1. Introduction

One of our major purposes of this research is to investigate properties of demand functions for goods and financial assets when agents face multiple budgets in incomplete financial asset markets. In trying to do so, we have found that Slutsky's method of income compensation in contrast to Hick's method is a more convenient rout in obtaining properties of demand functions since multiple budgets are present. But analytically Slutsky's method is more difficult to handle because a similar property such as the concavity of the minimum expenditure function is not easily available for this case. As we shall point out in this paper, the optimum value function associated with the Slutskian method possesses only the local convexity property in contrast to the minimum expenditure function in the Hicksian method.

In order to facilitate such analyses in a more systematic fashion, we find it useful first to establish an interesting second derivative property of an optimum value function that may be regarded as a second derivative envelope property. We then show the usefulness of this second derivative envelope property by applying it to the derivation of well-known properties of the Hicksian compensated demand function, the Slutskian compensated demand function and the indirect utility function in the standard demand theory. Of course there is nothing new in these results we obtain in this section. What is new lies in the use of the second derivative envelope property in obtaining these results and furthermore in showing that we can obtain the usual results in sharper and more transparent fashion. Finally we apply this approach using the second derivative envelope property in the investigation of properties of demand functions for goods and financial assets when agents face multiple budgets in incomplete financial asset markets. We hope that the second derivative envelope property may become one of useful formulas in providing microeconomic properties to various economic models involving optimizing behavior of agents.

2. Properties of an Optimum Value Function

We shall begin our investigation by providing useful derivative properties of the optimum value function for a constrained optimization problem.

Let $f : X \times A \to R$ and $g : X \times A \to R^m$ be C^2 functions where X and A respectively are open subsets of R^m and R^k. Define the optimum value function $\varphi : A \to R$ by

$$\varphi(\alpha) \equiv \max_x \left\{ f(x,\alpha) \middle| g(x,\alpha) = 0_m \right\}$$

where $\alpha \in A$ defines a parameter for the optimization. Let the Lagrangean function associated with the above maximization problem be written by:

$$L(\lambda, x, \alpha) \equiv f(x,\alpha) + \lambda \bullet g(x,\alpha)$$

for $\lambda \in R^m$. We will assume throughout this paper that, for each parameter α, the maximization problem is well-defined, satisfies the second-order regularity condition to be stated below and it yields the C^1 solution function $\tilde{x}(\alpha)$ and the associated C^1 Lagrange multiplier function $\tilde{\lambda}(\alpha)$. We shall state the second-order regularity condition as follows.

Assumption. (The second-order regularity condition) Given $\alpha \in A$, let $x = \tilde{x}(\alpha)$ and $\lambda = \tilde{\lambda}(\alpha)$. Then $d_{xx}L(\lambda, x, \alpha)$ is negative definite on the kernel of the linear map $d_x g(x,\alpha)$, i.e., on the linear subspace $\left\{ v \in R^m \middle\| \left[d_x g(x,\alpha) \right] v = 0 \right\}$ where $d_x g(x,\alpha)$ and $d_{xx}L(\lambda, x, \alpha)$ respectively denote the first derivative of g with respect to x and the second derivative of L with respect to x.

By the envelope theorem in economics (see Samuelson [1947], Otani and El-Hodiri

[1987] and Otani [1999]), the first derivative of the optimum value function satisfies:

$$d_\alpha \varphi(\alpha) = d_\alpha L(\lambda, x, \alpha) = d_\alpha f(x,\alpha) + \lambda' d_\alpha g(x,\alpha)$$

where $d_\alpha L(\lambda, x, \alpha)$ and $d_\alpha f(x,\alpha)$ are both $(1\times k)$ vectors of the first derivatives of L and f with respect to α, $d_\alpha g(x,\alpha)$ is an $(m\times k)$ matrix of the first derivative of g with respect to α, and all derivatives are evaluated at $\lambda = \tilde{\lambda}(\alpha)$ and $x = \tilde{x}(\alpha)$.

Derivatives of $\tilde{\lambda}(\alpha)$ and $\tilde{x}(\alpha)$ satisfy the usual comparative static formula derived by the implicit differentiation of the first-order conditions:

$$H\begin{bmatrix} d\tilde{\lambda} \\ d\tilde{x} \end{bmatrix} = -\begin{bmatrix} d_\alpha g \\ d_{x\alpha} f + \lambda \bullet d_{x\alpha} g \end{bmatrix}$$

where $\lambda \bullet d_{x\alpha} g \equiv \sum_{j=1}^{m} \lambda_j d_{x\alpha} g^j$ and H is the bordered Hessian of the Lagrangean given as follows:

$$H \equiv \begin{bmatrix} 0_{mm} & d_x g \\ (d_x g)' & d_{xx} f + \lambda \bullet d_{xx} g \end{bmatrix}.$$

We rewrite the above comparative static formula as:

$\left(d_x g\right) d\tilde{x} = -d_\alpha g$, and

$$\left(d_x g\right)' d\tilde{\lambda} + \left(d_{xx} f + \lambda \bullet d_{xx} g\right) d\tilde{x} = -\left(d_{x\alpha} f + \lambda \bullet d_{x\alpha} g\right).$$

When we differentiate $d\varphi(\alpha)$ once more, we obtain that

$$d^2\varphi(\alpha) = \left(d_\alpha g\right)' d\tilde{\lambda} + \left(d_{\alpha x} f + \lambda \bullet d_{\alpha x} g\right) d\tilde{x} + \left(d_{\alpha\alpha} f + \lambda \bullet d_{\alpha\alpha} g\right)$$

Using the comparative static formula, we have that $\left(d_\alpha g\right)' = -\left(d\tilde{x}\right)' \left(d_x g\right)'$ and

$$\left(d_\alpha g\right)' d\tilde{\lambda} = -\left(d\tilde{x}\right)' \left(d_x g\right)' d\tilde{\lambda} = \left(d\tilde{x}\right)' \left(d_{xx} f + \lambda \bullet d_{xx} g\right)\left(d\tilde{x}\right) + \left(d\tilde{x}\right)' \left(d_{x\alpha} f + \lambda \bullet d_{x\alpha} g\right).$$

Substituting the above into $d^2\varphi$ yields

$$d^2\varphi(\alpha) = \left(d\tilde{x}\right)'\left(d_{xx}f + \lambda \bullet d_{xx}g\right)\left(d\tilde{x}\right) + \left(d\tilde{x}\right)'\left(d_{x\alpha}f + \lambda \bullet d_{x\alpha}g\right)$$

$$+\left(d_{\alpha x}f + \lambda \bullet d_{\alpha x}g\right)\left(d\tilde{x}\right) + \left(d_{\alpha\alpha}f + \lambda \bullet d_{\alpha\alpha}g\right)$$

$$= \left(d\tilde{x}\right)'\left(d_{xx}L\right)\left(d\tilde{x}\right) + \left(d\tilde{x}\right)'\left(d_{x\alpha}L\right) + \left(d_{\alpha x}L\right)\left(d\tilde{x}\right) + d_{\alpha\alpha}L$$

$$= \begin{bmatrix} \left(d\tilde{x}\right)' & I_k \end{bmatrix} \begin{bmatrix} d_{xx}L & d_{x\alpha}L \\ d_{\alpha x}L & d_{\alpha\alpha}L \end{bmatrix} \begin{bmatrix} d\tilde{x} \\ I_k \end{bmatrix} = \left[d_\alpha\left(\tilde{x},\alpha\right)\right]'\left(d^2_{(x,\alpha)}L\right)\left[d_\alpha\left(\tilde{x},\alpha\right)\right]$$

where I_k denotes the $(k \times k)$ identity matrix and $d^2_{(x,\alpha)}L$ denotes the second derivative of L with respect to (x,α). Note that the above formula does not directly entail the second derivatives of $\tilde{x}(\alpha)$ and $\tilde{\lambda}(\alpha)$. Thus in this regard, the above result can be considered as a second derivative version of the envelope theorem. We state this result as a theorem.

Theorem 1. (The second derivative envelope property) The second derivative of the optimum value function satisfies the following formula:

$$d^2\varphi(\alpha) = \left(d\tilde{x}\right)'\left(d_{xx}L\right)d\tilde{x} + \left(d\tilde{x}\right)'\left(d_{x\alpha}L\right) + \left(d_{\alpha x}L\right)d\tilde{x} + d_{\alpha\alpha}L$$

$$= \begin{bmatrix} \left(d\tilde{x}\right)' & I_k \end{bmatrix} \begin{bmatrix} d_{xx}L & d_{x\alpha}L \\ d_{\alpha x}L & d_{\alpha\alpha}L \end{bmatrix} \begin{bmatrix} d\tilde{x} \\ I_k \end{bmatrix} = \left[d_\alpha\left(\tilde{x},\alpha\right)\right]'\left(d^2_{(x,\alpha)}L\right)\left[d_\alpha\left(\tilde{x},\alpha\right)\right]$$

where (λ, x) is evaluated at $\left(\tilde{\lambda}(\alpha), \tilde{x}(\alpha)\right)$.

For the rest of the paper, we will show how the above formula can be usefully applied to various cases of demand theory. In doing so, we shall maintain the same set of assumptions we have indicated in this section for each application.

3. Applications to the Standard Theory of Demand

In order to appreciate the usefulness of the above result, we will first look at the standard theory of demand. In particular we will show how the above second derivative envelope property can be successfully applied in obtaining properties of the Hicksian compensated demand function, the Slutskian compensated demand function and the indirect utility function.

3. 1 Hicksian compensated demand

The minimum expenditure function $m(p,\upsilon)$ in the standard demand theory is defined as follows.

$$m(p,\upsilon) \equiv \max_{x} \left\{ p \bullet x \middle| u(x) \geq \upsilon, x \in P^{\ell} \right\}$$

where P^{ℓ} is the interior of R_{+}^{ℓ}, $u : P^{\ell} \rightarrow R$ is a C^2 utility function, $\upsilon \in R$ is a utility level and $p \in P^{\ell}$ is a price vector. We note that the second-order regularity condition for this problem is that the utility function satisfies the regular strict quasi-concavity, i.e., $d^2u(x)$ is negative definite on the subspace $Ker\left[du(x)\right]$. We will assume that this minimization problem yields a solution in P^{ℓ}. The Lagrangean function of this minimization problem can be written as:

$$L(\mu, x; p, \upsilon) \equiv p \bullet x + \mu\left(\upsilon - u(x)\right).$$

Let $g(p,\upsilon)$ be the Hicksian compensated demand function and $\mu(p,\upsilon)$ be the associated Lagrange multiplier. (We apologize the use of the notation g which was used to indicate a constraint in the previous section.) As in the last section, both $g(p,\upsilon)$ and $\mu(p,\upsilon)$ are assumed to be C^1. Then by the envelope theorem, we have that:

$d_p m(p,\upsilon) = \left[g(p,\upsilon)\right]'$, and

$$d_\upsilon m(p,\upsilon) = \mu(p,\upsilon).$$

It is well-known and immediate from the above result that the second derivative $d_{pp}m$ and $d_{p\upsilon}m$ are given as follows:

$$d_{pp}m(p,\upsilon) = d_p g(p,\upsilon), \text{ and}$$

$$d_{p\upsilon}m(p,\upsilon) = d_\upsilon g(p,\upsilon).$$

Thus the second derivative $d_{pp}m$ is the derivative of the Hicksian compensated demand function with respect to p called the Slutsky matrix.

For this expenditure minimization problem, our formula on the second derivative of the optimum value function yields

$$\begin{bmatrix} d_{pp}m & d_{p\upsilon}m \\ d_{\upsilon p}m & d_{\upsilon\upsilon}m \end{bmatrix} = \begin{bmatrix} (d_p g)' & I_\ell & 0_{\ell\times 1} \\ (d_\upsilon g)' & 0_{1\times\ell} & 1 \end{bmatrix} \begin{bmatrix} -\mu d^2 u & I_\ell & 0_{\ell\times(\ell+1)} \\ I_\ell & 0_{\ell\times\ell} & 0_{\ell\times(\ell+1)} \\ 0_{(\ell+1)\times\ell} & 0_{(\ell+1)\times\ell} & 0_{(\ell+1)\times(\ell+1)} \end{bmatrix} \begin{bmatrix} d_p g & d_\upsilon g \\ I_\ell & 0_{\ell\times 1} \\ 0_{1\times\ell} & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} -\mu(d_p g)'(d^2 u)(d_p g) + (d_p g)' + (d_p g) & -\mu(d_p g)'(d^2 u)(d_\upsilon g) + (d_\upsilon g) \\ -\mu(d_\upsilon g)'(d^2 u)(d_p g) + (d_\upsilon g)' & -\mu(d_\upsilon g)'(d^2 u)(d_\upsilon g) \end{bmatrix}.$$

If we compare these two expressions on the second derivatives of the minimum expenditure function, we find that

$$d_{pp}m(p,\upsilon) = d_p g(p,\upsilon) = \left[d_p g(p,\upsilon)\right]' = \mu(d_p g)'(d^2 u)(d_p g), \text{ and}$$

$$(d_p g)'(d^2 u)(d_\upsilon g) = 0_{\ell\times 1}.$$

We may state these as the following lemma.

Lemma 1: The second derivative of the minimum expenditure function $m(p,\upsilon)$ satisfies

$$\begin{bmatrix} d_{pp}m & d_{p\upsilon}m \\ d_{\upsilon p}m & d_{\upsilon\upsilon}m \end{bmatrix} == \begin{bmatrix} \mu\left(d_p g\right)'\left(d^2u\right)\left(d_p g\right) & \left(d_\upsilon g\right) \\ \left(d_\upsilon g\right)' & -\mu\left(d_\upsilon g\right)'\left(d^2u\right)\left(d_\upsilon g\right) \end{bmatrix}.$$

As we can see from Lemma 1, the Slutsky matrix $d_{pp}m = d_p g$ can be expressed directly in terms of the second derivative d^2u of the utility function. We believe that this may be the first time such a relation is exhibited. The following technical facts are also useful to state as a lemma.

Lemma 2: (i) $d_{p_i} g \in Ker\left[du(x)\right] = \left[p\right]^\perp$ and $d_\upsilon g \notin \left[p\right]^\perp$, (ii) $rank\left(d_p g\right) = \ell - 1$, and (iii) if $\eta \in \left[p\right]^\perp$ and $\eta \neq 0_\ell$, then $\left[d_p g\right]\eta \neq 0$.

Proof:

(i) Since $g(p,\upsilon)$ is homogeneous of degree 0 in p and $d_p g$ is symmetric, $\left(d_p g\right)p = \left(d_p g\right)' p = 0_\ell$. From the first-order condition of the minimization, we have that $\left[du\right] = \left[p\right]$. Also since $u\left[g(p,\upsilon)\right] = \upsilon$, we get that $\left(du\right)\left(d_\upsilon g\right) = 1$. Thus we must have $d_{p_i} g \in Ker\left[du(x)\right] = \left[p\right]^\perp$ and $d_\upsilon g \notin \left[p\right]^\perp$.

(ii) and (iii) From the constraint $\upsilon - u(x) = 0$ and the first-order conditions $p - \mu du(x) = 0$, we can obtain the following system of comparative static equations

$$\begin{bmatrix} 0 & du \\ \left(du\right)' & d^2u \end{bmatrix} \begin{bmatrix} (1/\mu)d_p\mu & (1/\mu)d_\upsilon\mu \\ d_p g & d_\upsilon g \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ (1/\mu)I_\ell & 0_\ell \end{bmatrix}$$

From this we can assert that the rank of $\left[d_p g \quad d_\upsilon g\right]$ must be ℓ and hence the rank of $d_p g$ must be $(\ell - 1)$. The assertion in (iii) then follows.□

By the second-order regularity condition, we can conclude that for every $\eta \in \left[p\right]^\perp$,

$\eta \neq 0_\ell$,

$$\eta'\left[d_{pp}m\right]\eta = \eta'\left[d_p g\right]\eta = \mu\eta'\left(d_p g\right)'\left[d^2u\right]\left(d_p g\right)\eta = -\eta'\left(d_p g\right)'\left[d_{xx}L\right]\left(d_p g\right)\eta < 0.$$

We may state this as the following proposition.

Proposition 1. The Slutsky matrix $d_p g\left(p,\upsilon\right)$ is negative definite on the subspace $[p]^\perp$.

As is usually done, it is easy to show that the Slutsky matrix $d_p g(p,\upsilon)$ is negative semi-definite from the concavity and the differentiability of $m(p,\upsilon)$ with respect to p. But our result provides us with a sharper property of the Slutsky matrix, in particular any $(\ell-1)$ principal minor submatrix of $d_p g(p,\upsilon)$ is nonsingular. This property is usually obtained by using the inversion formula of a partitioned matrix. But we are able to show it more directly from the second-order condition because we are able to express the Slutsky matrix directly in terms of d^2u. (See Otani and El-Hodiri [1987] for an alternative approach to get this result.)

3. 2 Slutsky's Method of Compensation and the Slutskian Compensated Demand

Our next application concerns Slutsky's method of compensation and the resulting Slutskian compensated demand function. (See Katzner [1970], Section 3.4, and Otani and El-Hodiri [1987], Chapter 2 Section 3, for Hicks' and Slutsky's methods of compensation.) Given $x^0 \gg 0$, the Slutskian compensated demand function is defined by

$$G(p,x^0) \equiv h(p, p\cdot x^0)$$

where $h(p,w)$ denotes the ordinary utility maximizing demand function. Let $x^0 = G(p^0,x^0)$. Then the Slutskian compensated demand function can be considered as a solution to the utility maximizing problem with a budget constraint $p\cdot(x^0-x)\geq 0$. Let the optimum value function for this problem be defined by:

$$\varphi(p,x^0)\equiv\max_x\left\{u(x)\middle|\, p\cdot(x^0-x)\geq 0\right\}$$

and the Lagrangean function of this maximization problem be written as:

$$L(\gamma,x;p,x^0)\equiv u(x)+p\cdot(x^0-x)\,.$$

We let the Lagrange multiplier function associated with the above maximization be denoted by $\gamma(p,x^0)$. Then the second derivatives of φ with respect to p evaluated at p^0 becomes

$$d_{pp}\varphi(p^0,x^0)=-\gamma^0 d_p G(p^0,x^0)$$

where $\gamma^0=\gamma(p^0,x^0)$. Thus the matrix $d_pG(p,x^0)$ is symmetric. If we apply the second derivative formula to $d_{pp}\varphi$, we obtain that

$$d_{pp}\varphi=\left(d_pG\right)'\left(d^2u\right)\left(d_pG\right)-\gamma\left(d_pG\right)'-\gamma\left(d_pG\right).$$

Evaluating at p^0 and using the symmetry of d_pG yield

$$d_pG(p^0,x^0)=\left(1/\gamma^0\right)\left(d_pG(p^0,x^0)\right)'\left(d^2u(x^0)\right)\left(d_pG(p^0,x^0)\right)$$

As in the Hicksian case, we can show the following Lemma and Proposition whose proofs are omitted since they are similar to those of Lemma 2 and Proposition 1.

Lemma 3: (i) $d_{p_i}G\in Ker\left[du(x^0)\right]=\left[p^0\right]^{\perp}$ and $d_{\upsilon}G\notin\left[p^0\right]^{\perp}$, (ii) $rank\left(d_pG\right)=\ell-1$, and (iii) if $\eta\in\left[p^0\right]^{\perp}$ and $\eta\neq 0_{\ell}$, then $\left[d_pG\right]\eta\neq 0$ where derivatives of G are evaluated at (p^0,x^0).

Proposition 2. The matrix $d_p G(p^0, x^0)$ is negative definite on the subspace $\left[p^0\right]^\perp$.

In our framework, it is well known and easy to show that $x = g(p,\upsilon)$ if and only if $x = G(p,x)$ and $\upsilon = \varphi(p,x)$. Also as is well-known, if $x = g(p,\upsilon)$, then

$$G(p,x) = h(p, p\cdot x) = h\big(p, p\cdot g(p,\upsilon)\big) = h\big(p, m(p,\upsilon)\big) = g(p,\upsilon).$$

Thus we have $d_p g(p,\upsilon) = d_p G(p,x)$ provided that $x = g(p,\upsilon)$.

It should be noted that unlike the minimum expenditure function which is concave in p, the second derivative of φ is positive definite on $\left[p^0\right]^\perp$ only when it is evaluated at (p^0, x^0). Thus φ is a convex function only locally around (p^0, x^0) and, without using the second derivative envelope property, it would not be so easy to assert the negative semi-definiteness of the Slutsky matrix via the Slutskian compensated demand function. As we show in Section 4, when there are multiple budgets, the Hicksian method can not be directly adopted, but the Slutskian method becomes directly applicable. In such an occasion, our approach will provide a useful avenue as we show in Section 4.

3. 3 Properties of the Indirect Utility Function

Consider the indirect utility function defined as follows.

$$\upsilon(p,w) = \max_x \left\{u(x) \middle| w - p\cdot x \geq 0\right\}.$$

The Lagrangean function for this problem can be written as

$$L(\lambda, x; p, w) \equiv u(x) + \lambda(w - p\cdot x).$$

Let $h(p,w)$ be the solution to this maximization problem and $\lambda(p,w)$ be the associated Lagrange multiplier function.

Applying the second derivative formula to $d_{ww}\upsilon$ yields

$$d_{ww}\upsilon = \left(d_w h\right)' \left(d^2 u\right)\left(d_w h\right)$$

By the budget condition, we have that $p \cdot d_w h = 1$ and hence $d_w h \notin Ker\left[du(x)\right] = \left[p\right]^{\perp}$. Therefore the second-order regularity condition can not be applied to get the negativity of $d_{ww}\upsilon$ if the utility function satisfies merely regular strict quasi-concavity. Only when the utility function is regularly strict concave, i.e., $d^2u(x)$ is negative definite, we can assert that $d_{ww}\upsilon < 0$. One interesting case to note is when the utility function is homogeneous of degree one, then we can show that $d_{ww}\upsilon = 0$.

4. Demand Theory with Financial Assets and Multiple Budgets

In this section, we shall analyze the decision problem of a consumption agent in an economy with financial assets and incomplete markets so that the agent will face multiple budget constraints. In this economy, there are two periods, period 0 and period 1. There is no uncertainty in period 0, and the certainty state in period 0 will be denoted by $s = 0$. In period 1, there are S uncertain states denoted by $s = 1, \ldots, S$. There are L physical goods for each state $s = 0, 1 \ldots, S$ and $x(s) \in P^L$ denotes a consumption vector at state s. The agent is assumed to have a C^2 utility function $u : X \to R$ where $X = P^{\ell}$ and $\ell \equiv L(S+1)$. Then the decision problem of the agent is to maximize $u(\mathbf{x}) = u\left(x(0), x(1), \ldots, x(S)\right)$ subject to the following multiple budget constraints:

$p(0) \cdot x(0) + q \cdot \theta \leq w(0)$, and

$p(s) \cdot x(s) \leq w(s) + a(s) \cdot \theta \quad (s = 1, \ldots, S)$

where $\mathbf{x} \equiv \left(x(0), x(1), \ldots, x(S)\right)$, $x(s) \in P^{\ell}$, $p(s) \in P^{\ell}$ is a vector of spot good prices in state s, $w(s) \in P$ denotes the nominal nonfinancial income in state s in an accounting unit, $q \in R^J$ is a vector of the prices of financial assets, $\theta = (\theta_1, \ldots, \theta_J)' \in R^J$ where

$\theta_j \in R$ denotes the holding of the j-th financial asset, $a(s) \equiv \left(a_1(s),\ldots,a_J(s)\right)'$, and $a_j(s)$ indicates a promise of the delivery of the accounting unit when state s occurs for the j-th financial asset. Thus the j-th financial asset can be characterized by its return vector $a_j \equiv \left(a_j(1),\ldots,a_j(S)\right)'$. We will assume the possibility of incomplete markets so that $J \le S$. The $(S \times J)$ return matrix will be denoted by

$$\mathbf{A} \equiv \left[a_1,\ldots,a_J\right] = \left[a(1),\ldots,a(S)\right]'$$

and we write

$$\mathbf{A}_0 \equiv \left[a(0),a(1),\ldots,a(S)\right]' = \left[a(0),\mathbf{A}'\right]'$$

where $a(0) \equiv -q$.

We will decompose the above decision problem into two sub-problems, the first problem on the choice of real consumption vectors, and the second on the choice of financial assets so that state incomes can be adjusted among states through the holding of financial assets.

Let $\mathbf{b} \equiv \left(b(0),b(1),\ldots,b(S)\right) \in P^{S+1}$ be a vector of state incomes and $S_0 \equiv \left\{0,1,\ldots,S\right\}$. Then the first problem is to maximize $u(\mathbf{x})$ subject to the multiple budget constraints as follows:

$$p(s)\cdot x(s) \le b(s) \quad (s \in S_0)$$

The maximum value function for this problem can be defined by:

$$\upsilon(\mathbf{p},\mathbf{b}) = \max_{\mathbf{x}} \left\{u(\mathbf{x}) \middle| \left(\forall s \in S_0\right) p(s)\cdot x(s) \le b(s)\right\}$$

where $\mathbf{p} \equiv \left(p(0)',\ldots,p(S)'\right)'$ and $\mathbf{b} \equiv \left(b(0),\ldots,b(S)\right)'$.

Let $\mathbf{w} \equiv \left(w(0),\ldots,w(S)\right)'$. Then the second problem is to choose $\theta \in R^J$ to maximize

$$\upsilon\left(\mathbf{p},\mathbf{w}+\mathbf{A}_0\theta\right)=\upsilon\left(\mathbf{p},w(0)+a(0)\cdot\theta,\ldots,w(S)+a(S)\cdot\theta\right).$$

This is basically a problem of portfolio choice to allocate state incomes through financial assets.

We start out with the analysis of the first problem. Let the Lagrangean function of the first problem be given by:

$$L\left(\lambda,\mathbf{x},\mathbf{p},\mathbf{b}\right)=u(\mathbf{x})+\sum_{s\in S_0}\lambda(s)\left(b(s)-p(s)\cdot x(s)\right).$$

Then the first-order conditions are given as follows:

$$d_{x(s)}u(\mathbf{x})-\lambda(s)p(s)'=0 \quad (s\in S_0)$$

Let the demand function for consumption goods for state s be denoted by $h_s(\mathbf{p},\mathbf{b})$ and $h(\mathbf{p},\mathbf{b})\equiv\left[h_0(\mathbf{p},\mathbf{b})',\ldots,h_S(\mathbf{p},\mathbf{b})'\right]'$. We will be interested in decomposing a price effect on demand into the substitution effect and the income effect as in the standard demand theory. The Hicksian approach to income compensation uses the expenditure minimization along an indifference surface. But with the presence of multiple budgets in this case, the expenditure to be minimized becomes ambiguous. But fortunately the income compensation scheme by Slutsky poses no problem for this case even with multiple budgets and our approach in 3.2 will turn out to be very useful for our problem in this section.

Let us define the Slutskian compensated demand function as follows:

$$G\left(\mathbf{p},\mathbf{x}^0\right)\equiv h\left(\mathbf{p},p(0)\cdot x^0(0),\ldots,p(S)\cdot x^0(S)\right)$$

This function can be characterized as a solution to the utility maximization problem with a vector of state incomes $\mathbf{b}=\left(p(0)\cdot x^0(0),\ldots,p(S)\cdot x^0(S)\right)$. The optimum value function and the associated Lagrangean for this maximization problem is given respectively by:

$$\varphi(\mathbf{p},\mathbf{x}^0) \equiv \max_{\mathbf{x}} \left\{ u(\mathbf{x}) \middle| \left(\forall s \in S_0\right) p(s)\cdot\left(x^0(s)-x(s)\right) \right\}, \text{ and}$$

$$L(\gamma,\mathbf{x};\mathbf{p},\mathbf{x}^0) \equiv u(\mathbf{x}) + \sum_{s\in S_0} \gamma_s p(s)\cdot\left(x^0(s)-x(s)\right).$$

Let $\gamma(\mathbf{p},\mathbf{x}^0) \equiv \left(\gamma_0(\mathbf{p},\mathbf{x}^0),\ldots,\gamma_S(\mathbf{p},\mathbf{x}^0)\right)$ be the associated Lagrange multiplier function. We will assume that $\mathbf{x}^0 = G(\mathbf{p}^0,\mathbf{x}^0)$.

By the envelope theorem, the first derivative of φ with respect to $p(s)$ is given as follows:

$$d_{p(s)}\varphi(\mathbf{p},\mathbf{x}^0) = \gamma_s(\mathbf{p},\mathbf{x}^0)\left(x^0(s)-G_s(\mathbf{p},\mathbf{x}^0)\right)'$$

By differentiating $d_{p(s)}\varphi$ once more with respect to $p(s')$, we obtain that:

$$d_{p(s)p(s')}\varphi(\mathbf{p},\mathbf{x}^0) = \left[d_{p(s')}\gamma_s(\mathbf{p},\mathbf{x}^0)\right]'\left[x^0(s)-G_s(\mathbf{p},\mathbf{x}^0)\right]' - \gamma_s(\mathbf{p},\mathbf{x}^0)d_{p(s')}G_s(\mathbf{p},\mathbf{x}^0).$$

Evaluating the above at $\mathbf{p}^0$ yields

$$d_{p(s)p(s')}\varphi(\mathbf{p}^0,\mathbf{x}^0) = -\gamma_s^0 d_{p(s')}G_s(\mathbf{p}^0,\mathbf{x}^0)$$

where $\gamma_s^0 \equiv \gamma_s(\mathbf{p}^0,\mathbf{x}^0)$. Thus

$$-d_{\mathbf{pp}}\varphi(\mathbf{p}^0,\mathbf{x}^0) = \Gamma^0 d_{\mathbf{p}}G(\mathbf{p}^0,\mathbf{x}^0)$$

where Γ^0 is an $(\ell\times\ell)$ diagonal matrix with blocks $\gamma_s^0 I_L$ ($s=0,\ldots,S$) as follows:

$$\Gamma^0 = \begin{bmatrix} \gamma_0^0 I_L & \cdots & 0_{L\times L} \\ \vdots & \ddots & \vdots \\ 0_{L\times L} & \cdots & \gamma_S^0 I_L \end{bmatrix}.$$

Thus $\Gamma^0 d_{\mathbf{p}}G(\mathbf{p}^0,\mathbf{x}^0)$ becomes a symmetric matrix.

If we apply the second derivative envelope property to the optimum value function φ, we can obtain that:

$$d_{\mathbf{pp}}\varphi = \left(d_{\mathbf{p}}G\right)'\left[d^2u\right]\left(d_{\mathbf{p}}G\right) - \left(d_{\mathbf{p}}G\right)'\Gamma - \Gamma\left(d_{\mathbf{p}}G\right).$$

Evaluating at $\mathbf{p}^0$ yields

$$-d_{\mathbf{pp}}\varphi(\mathbf{p}^0,\mathbf{x}^0) = \Gamma^0 d_{\mathbf{p}}G(\mathbf{p}^0,\mathbf{x}^0) = \left(d_{\mathbf{p}}G(\mathbf{p}^0,\mathbf{x}^0)\right)'\left[d^2u(\mathbf{x}^0)\right]\left(d_{\mathbf{p}}G(\mathbf{p}^0,\mathbf{x}^0)\right) \text{ and}$$

$$-d_{p(s)p(s)}\varphi(\mathbf{p}^0,\mathbf{x}^0) = \gamma_s^0 d_{p(s)}G(\mathbf{p}^0,\mathbf{x}^0) = \left(d_{p(s)}G(\mathbf{p}^0,\mathbf{x}^0)\right)'\left[d^2u(\mathbf{x}^0)\right]\left(d_{p(s)}G(\mathbf{p}^0,\mathbf{x}^0)\right)$$

Since the Slutskian compensated demand function $G(\mathbf{p},\mathbf{x}^0)$ is homogeneous of degree zero in each $p(s)$ and in $\mathbf{p}$, we have that:

$\left[d_{p(s)}G(\mathbf{p},\mathbf{x}^0)\right]p(s) = 0_\ell$, and

$\left[d_{\mathbf{p}}G(\mathbf{p},\mathbf{x}^0)\right]\mathbf{p} = 0_\ell$.

As in Section 3.2, if $\eta \in \left[\mathbf{p}^0\right]^\perp$ and $\eta \neq 0_\ell$, then

$\left[d_{\mathbf{p}}G(\mathbf{p}^0,\mathbf{x}^0)\right]\eta \neq 0_\ell$, and

if $\eta_s \in \left[p(s)\right]^\perp$ and $\eta_s \neq 0_L$, then

$\left[d_{p(s)}G(\mathbf{p}^0,\mathbf{x}^0)\right]\eta_s \neq 0_\ell$.

Thus we can conclude that $\Gamma^0 d_{\mathbf{p}}G(\mathbf{p}^0,\mathbf{x}^0)$ is negative definite on $\left[\mathbf{p}^0\right]^\perp$ and since $\gamma_s^0 > 0$, $d_{p(s)}G(\mathbf{p}^0,\mathbf{x}^0)$ is negative definite on $\left[p(s)\right]^\perp$. Differentiating the definition of the Slutskian compensated demand function with respect to $p(s)$ yields the Slutsky equation as follows:

$$d_{p(s)}G(\mathbf{p}^0,\mathbf{x}^0) = d_{p(s)}h(\mathbf{p}^0,\mathbf{b}^0) + \left[d_{b(s)}h(\mathbf{p}^0,\mathbf{b}^0)\right]'\left(x^0(s)\right)'$$

where $b^0(s) \equiv p^0(s)\cdot x^0(s)$. We can summarize these findings as follows:

Proposition 3. (i) $\Gamma^0 d_{\mathbf{p}} G(\mathbf{p}^0, \mathbf{x}^0)$ is symmetric and negative definite on $\left[\mathbf{p}^0\right]^{\perp}$ and $d_{p(s)} G(\mathbf{p}^0, \mathbf{x}^0)$ is symmetric and negative definite on $\left[p(s)\right]^{\perp}$, (ii) the following Slutsky equation holds

$$d_{p(s)} G(\mathbf{p}^0, \mathbf{x}^0) = d_{p(s)} h(\mathbf{p}^0, \mathbf{b}^0) + \left[d_{b(s)} h(\mathbf{p}^0, \mathbf{b}^0)\right]' \left(x^0(s)\right)'$$

where $b^0(s) \equiv p^0(s) \cdot x^0(s)$.

We now turn to the portfolio choice problem of choosing θ to maximize the indirect utility function $\psi\left(\theta; \mathbf{p}, \mathbf{w}, \mathbf{A}_0\right) \equiv \upsilon\left(\mathbf{p}, \mathbf{w} + \mathbf{A}_0 \theta\right)$ with respect to θ. By the envelope theorem, the first derivatives of the indirect utility function $\upsilon\left(\mathbf{p}, \mathbf{b}\right)$ are given as follows:

$$d_{p(s)} \upsilon\left(\mathbf{p}, \mathbf{b}\right) = -\lambda(s) \left[h_s\left(\mathbf{p}, \mathbf{b}\right)\right]', \text{ and}$$

$$d_{b(s)} \upsilon\left(\mathbf{p}, \mathbf{b}\right) = \lambda(s)$$

By applying the second derivative envelope property to υ, we can obtain the second derivative of υ with respect to $\mathbf{b}$ in a similar fashion as in Section 3.3 to obtain:

$$d_{\mathbf{bb}} \upsilon\left(\mathbf{p}, \mathbf{b}\right) = d_{\mathbf{bb}} \upsilon\left(\mathbf{p}, \mathbf{b}\right) + d_{\mathbf{bb}} \upsilon\left(\mathbf{p}, \mathbf{b}\right) - \left[d_{\mathbf{b}} h\left(\mathbf{p}, \mathbf{b}\right)\right]' \left(d^2 u(\mathbf{x})\right) \left[d_{\mathbf{b}} h\left(\mathbf{p}, \mathbf{b}\right)\right].$$

Thus we have that:

$$d_{\mathbf{bb}} \upsilon\left(\mathbf{p}, \mathbf{b}\right) = \left[d_{\mathbf{b}} h\left(\mathbf{p}, \mathbf{b}\right)\right]' \left(d^2 u(\mathbf{x})\right) \left[d_{\mathbf{b}} h\left(\mathbf{p}, \mathbf{b}\right)\right].$$

Then

$$d_{\theta} \psi\left(\theta; \mathbf{p}, \mathbf{w}, \mathbf{A}_0\right) \equiv d_{\mathbf{b}} \upsilon\left(\mathbf{p}, \mathbf{w} + \mathbf{A}_0 \theta\right) \mathbf{A}_0, \text{ and}$$

$$d_{\theta\theta} \psi\left(\theta; \mathbf{p}, \mathbf{w}, \mathbf{A}_0\right) \equiv A_0' d_{\mathbf{bb}} \upsilon\left(\mathbf{p}, \mathbf{w} + \mathbf{A}_0 \theta\right) \mathbf{A}_0 = \mathbf{A}_0' \left[d_{\mathbf{b}} h\left(\mathbf{p}, \mathbf{b}\right)\right]' \left(d^2 u(\mathbf{x})\right) \left[d_{\mathbf{b}} h\left(\mathbf{p}, \mathbf{b}\right)\right] \mathbf{A}_0$$

where $\mathbf{x} = h(\mathbf{p},\mathbf{b})$ and $\mathbf{b} = \mathbf{w} + \mathbf{A}_0\theta$.

If we differentiate the budget identities $p(s)\cdot h_s(\mathbf{p},\mathbf{b}) = b(s)$ with respect to $b(s')$, we can obtain that:

$$p(s)'\left[d_{b(s')}h(\mathbf{p},\mathbf{b})\right] = \delta_{ss'}$$

where $\delta_{ss'} = 0$ if $s \neq s'$ and $\delta_{ss'} = 1$ if $s = s'$. Thus we have that:

$$B(\mathbf{p})\left[d_{\mathbf{b}}h(\mathbf{p},\mathbf{b})\right] = I_{S+1}$$

where

$$B(\mathbf{p}) \equiv \begin{bmatrix} p(0)' & 0 & \cdots & 0 \\ 0 & p(1)' & \vdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & 0 & p(S)' \end{bmatrix}.$$

Using the above relation, we can assert that $rank\left[d_{\mathbf{b}}h(\mathbf{p},\mathbf{b})\right] = S+1$. Now if $rank\left[\mathbf{A}_0\right] = J$, then we have that $rank\left[d_{\mathbf{b}}h(\mathbf{p},\mathbf{b})\mathbf{A}_0\right] = J$. Hence for any $\eta \in R^J$, $\left[d_{\mathbf{b}}h(\mathbf{p},\mathbf{b})\mathbf{A}_0\right]\eta = 0_\ell$ implies that $\eta = 0_J$, i.e., for any $\eta \in R^J$, if $\eta \neq 0_J$, then $\left[d_{\mathbf{b}}h(\mathbf{p},\mathbf{b})\mathbf{A}_0\right]\eta \neq 0_\ell$. Therefore if $d^2u(\mathbf{x})$ is negative definite and if $rank\left[\mathbf{A}_0\right] = J$, then for any $\eta \in R^J$, $\eta \neq 0_J$,

$$\eta'\left[d_{\theta\theta}\psi(\theta;\mathbf{p},\mathbf{w},\mathbf{A}_0)\right]\eta = \eta'\mathbf{A}_0'\left[d_{\mathbf{b}}h(\mathbf{p},\mathbf{b})\right]'\left(d^2u(\mathbf{x})\right)\left[d_{\mathbf{b}}h(\mathbf{p},\mathbf{b})\right]\mathbf{A}_0\eta < 0,$$

i.e., $d_{\theta\theta}\psi(\theta;\mathbf{p},\mathbf{w},\mathbf{A}_0)$ is negative definite. Hence the function ψ is strictly concave in θ. We state this result as a proposition.

Proposition 4. If the utility function is regularly strict concave, i.e., if $d^2u(\mathbf{x})$ is negative definite and if $rank\left[\mathbf{A}_0\right] = J$, then $d_{\theta\theta}\psi(\theta;\mathbf{p},\mathbf{w},\mathbf{A}_0)$ is negative definite and the function ψ is regularly strict concave in θ.

Therefore if the condition of this proposition is satisfied, then the portfolio choice problem yields a unique solution denoted by $\theta(\mathbf{p}, \mathbf{w}, \mathbf{A}_0)$ which is a portfolio demand function. The derivatives of $\theta(\mathbf{p}, \mathbf{w}, \mathbf{A}_0)$ with respect to $\mathbf{p}$ and $\mathbf{w}$ are given by the implicit differentiation rule as follows:

$d_{\mathbf{p}}\theta = -(d_{\theta\theta}\psi)^{-1}\,\mathbf{A}_0{}' d_{\mathbf{bp}}\upsilon$, and

$d_{\mathbf{w}}\theta = -(d_{\theta\theta}\psi)^{-1}\,\mathbf{A}_0{}' d_{\mathbf{bb}}\upsilon$.

These properties of the portfolio demand function can be applied in a straightforward manner to show the maximal rank condition of a certain mapping in the literature of real indeterminacies with financial assets provided that the utility function is regularly strict concave. (For example, the mapping g of Lemma 3 in Geanakopolos and Mas-Colell [1989, p.30] and the mapping F in Werner [1990, p.229].)

5. Conclusion

In this paper, we have derived a useful second derivative formula of the optimum value function which may be regarded as a second derivative envelope property. First we have shown that this formula can be applied fruitfully to various situations of the standard demand theory yielding sharper results in a straightforward manner. Particularly we have shown how we can use Slutsky's method of compensation in obtaining properties of the Slutsky matrix. Secondly we have investigated an approach to demand theory when a consumption agent faces multiple budgets and financial assets, particularly in incomplete markets. We have shown that, with the presence of multiple budgets, Slutsky's method of compensation becomes quite useful in establishing properties of substitution effects analogous to the standard demand theory. We have also shown that the second derivative envelope property of the optimum value function can be applied nicely in obtaining properties of the portfolio demand function arising in general equilibrium with financial assets. We believe that our results will be useful in providing microeconomic properties to various economic models involving optimizing behavior of agents.

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