The Structure of Equilibrium Consumption Allocations of a Lindahl Strategic Game

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[Abstract]

In this paper we investigates properties of the set of consumption allocations resulting from non-cooperative strategic interactions of agents who try to manipulate their demand behaviors to improve their well-beings in an economy with a public good and the Lindahl allocation mechanism. In particular we shall show that strategic externalities result in the real indeterminacy of allocations and the dimension of indeterminacy for the public good economy will stay large for a large economy and will be larger by the number of contributing agent types compared with the private good economy.

Keyword: public good, Lindahl equilibrium, strategic manipulation, existence
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1. Introduction

This is a sequel to our paper Otani (2001 a) in which the existence of a strategic equilibrium in a Lindahl strategic game is investigated. In this paper we investigates properties of the set of consumption allocations resulting from non-cooperative strategic interactions of agents who try to manipulate their demand behaviors to improve their well-beings in an economy with a public good and the Lindahl allocation mechanism.

As in Otani and Sicilian (1982, 1990) and Otani (1996) which investigate a model of a private goods economy with the Walrasian mechanism, we shall be interested in finding properties of the set of strategic or Cournot-Nash (C-N) equilibrium allocations, instead of those of incentive compatibility. The incentive compatibility problem as in Roberts and Postlewaite (1976) for private goods economies and Roberts (1976) for economies with public goods is mainly concerned with whether or not any agent will deviate from correctly revealing his/her true demand (or preferences) while other agents correctly reveal their true demand (or preferences). But our concern in this paper as well as in Otani and Sicilian (1982, 1990) and Otani (1996) is to answer questions on what would be the aggregate effects of strategic externalities resulting from manipulations of demand among agents. In particular, we shall investigate the structure and the size of strategic Lindahl equilibrium allocations whose existence is established in Otani (2001 a). We shall answer the question on limiting properties as the economy becomes larger in the paper to be followed [Otani (2001 c)].

1. The Model, Definitions, Assumptions and Basic Lemmas

In this section, we provide the description of our model and definitions and collect basic lemmas. For further explanations and the proofs of lemmas, readers should see Otani (2001 a). We shall generally consider an economy with 1 public good, l private goods and T consumption agents. We abuse a notation using T for the number of agents as well as for the set of all agents.

1.1. Production of the public good

The public good is produced by the application of inputs of private goods and the technology is assumed to be represented by the production function $F : R_+^l \to R$ denoted by $y = F(v)$. The production function is assumed to be continuous, strictly quasi-concave, $F(0) = 0$ and homogeneous of degree one. An input coefficient vector is denoted by $a_y \equiv v / y$. Given the price vector p of private goods, the unit cost function is defined as follows:

$$c(p) \equiv \min_{a_y} \left\{ p \cdot a_y \,\middle|\, F(a_y) \geq 1 \right\}$$

The minimizing vector of input coefficients as a function of prices of private goods will be denoted by: $a_y = a_y(p)$. Clearly $c(p) = p \cdot a_y(p)$. The profitability condition for the production of the public good is given as follows:

$$q - c(p) \leq 0, \text{ and}$$
$$y[q - c(p)] = 0$$

where $q \equiv \sum_{t \in T} q_t$ denotes the price of the public good with q_t being the contribution or the cost share of agent t.

1.2. Consumption agents

Consumption agent t is characterized by (u_t, ω_t) where $u_t : R_+^{l+1} \to R^*$ is his/her utility function and $\omega_t \in P^l$ with P indicating the set of strictly positive real numbers. The range R^* of utility functions is assumed to be the set of extended real numbers and thus possibly assume an infinite value at the boundary of the consumption set as in log-linear functions. We assume that the utility function is strictly quasi-concave, continuously differentiable on the positive orthant, $du_t = (d_x u_t, d_y u_t) \gg 0'_{l+1}$ and if $u_t(x_t, y) > u_t(\omega_t, 0)$, then $x_t \in P^l$. The budget map of agent t is defined as follows:

$$B_t(p, q_t) \equiv \left\{ (x_t, y) \in R_+^{l+1} \,\middle|\, p \cdot x_t + q_t \cdot y \leq p \cdot \omega_t \right\}$$

Let S_t be a parametric strategy space for agent t that is assumed to be a nonempty subset of a finite dimensional Euclidean space. We suppose that a given parameter $s_t \in S_t$ determines a strategic utility function $u_t(x_t, y, s_t)$ the agent uses. Strategic utility functions are assumed to be $u_t(\bullet, s_t): C(s_t) \to R^*$ where $C(s_t) \subseteq R_+^{l+1}$. A strategic utility function in turn determines a strategic demand function $f_t: P^l \times R_+ \times S_t \to R_+^l$ for private goods and a cost share function $k_t: P^l \times R_+ \times S_t \to R_+$ for the public good as follows:

$$\begin{aligned}&[f_t(p,y,s_t), k_t(p,y,s_t)] \\ &\equiv \left\{(x_t,q_t) \middle| (x_t,y) = \arg\max\left\{u_t(x_t,y,s_t) \middle| (x_t,y) \in B_t(p,q_t) \cap C(s_t)\right\}\right\}\end{aligned}$$

1.3. Definitions of equilibria

Let denote $S \equiv \prod_{t \in T} S_t$. A lindahl equilibrium given (a_y, c, s) and a consistent Lindahl equilibrium are defined as follows.

Definition 1. (a) $(p,y) \in P^l \times R_+$ is said to be a Lindahl equilibrium given $(a_y, c, s) \in R_+^l \times R_+ \times S$ if

(i) $\sum_{t \in T} \{f_t(p,y,s_t) - \omega_t\} - a_y y = 0$, and

(ii) $\sum_{t \in T} k_t(p,y,s_t) - c = 0$.

(b) $(p,y) \in P^l \times R_+$ is said to be a consistent Lindahl equilibrium given $s \in S$ if, in addition to (i) and (ii) above,

(iii) $a_y = a_y(p)$ and $c = c(p)$

holds.

The set of Lindahl equilibria given (a_y, c, s) will be denoted by $L(a_y, c, s)$ and the set of consistent Lindahl equilibria given s will be denoted by $L(s)$.

Definition 2. $(p^*, y^*, x^*, s^*) \in P^l \times R_+ \times R_+^{lT} \times S$ is said to be a strategic Lindahl equilibrium if (i) $(p^*, y^*) \in L(s^*)$ and $x_t^* = f_t(p^*, y^*, s_t^*)$ for every $t \in T$, and (ii) for every

$t \in T$ and for every $s_t \in S_t$, if $(p, y) \in L(a_y^*, c^*, s^* / s_t)$, $a_y^* = a_y(p^*)$, $c^* = c(p^*)$ and $x_t = f_t(p, y, s_t)$, then $u_t(x_t, y) \leq u_t(x_t^*, y^*)$.

Given strategies (s / s_τ) of other agents and given the input coefficient vector, agent τ faces the excess supply of private goods available to him/her as follows.

$$x_\tau(p, y, a_y, s / s_\tau) \equiv \sum_{t \in T} \omega_t - \sum_{t \neq \tau} f_t(p, y, s_t) - a_y y$$

which will be called the residual supply map of private goods for agent τ.

2.4. Strategy Space Contraction and Strategic Lindahl Equilibria

Let us consider the following family of utility functions $u(\bullet, s): C(s) \to R^*$ with a vector of $2l$ strategic parameters $s \equiv (\alpha, \beta, \gamma, \delta) \in R_+^{2l}$.

$$u(x, y, s) \equiv \sum_{i=1}^{l-1} \alpha_i \ln(x_i - \beta_i) + x_l + \gamma \ln(y + \upsilon) + \delta y$$

in which $C(s) \equiv \left\{ (x, y) \middle| x_i \geq \beta_i (i = 1, \cdots, l-1), x_l \geq 0, y \geq 0 \right\}$, $\upsilon > 0$ is a fixed non-strategic parameter common to all agents. This utility function generates the following system of demand functions for private goods and the cost share function for the public good as follows;

$$f_i(p, y, s) \equiv (\alpha_i / p_i) + \beta_i \qquad (i = 1, 2, \cdots, l-1)$$

$$f_l(p, y, s) \equiv \sum_{i=1}^{l-1} p_i(\omega_i - \beta_i) + (\omega_l - \sum_{i=1}^{l-1} \alpha_i) - \delta y - \gamma y / (y + \upsilon)$$

$$k(y, s) \equiv \delta + \gamma / (y + \upsilon)$$

where p_i denotes the price of private good i relative to that of good l provided that $x_l = f_l(p, y, s) \geq 0$. Let $S_t^0 \equiv A_t \times B_t \times G_t \times D_t$ where

$$A_t \equiv R_+^{l-1}, B_t \equiv \left\{ \beta \in R_t^{l-1} \middle| \beta_i \leq \omega_i \, (i = 1, \cdots, l-1) \right\}, G_t \equiv R_+, D_t \equiv R_+.$$

We shall assume that the above family of utility functions is in the strategy set of each agent.

Assumption 1. For each $t \in T$, $S_t^0 \subseteq S_t$ holds.

When $s_t \in S_t^0$ for every $t \neq \tau$, then the residual supply map for private goods for agent τ is given as follows.

$$x_{\tau i}(p, y, a_y, s/s_\tau) \equiv \omega_{\tau i} + w_{\tau i} - a_{yi} y - [(a_{\tau i}/p_i) + b_{\tau i}], \quad (i = 1, \cdots, l-1),$$

$$x_{\tau l}(p, y, a_y, s/s_\tau) \equiv \omega_{\tau l} + a_\tau + \left[g_\tau y/(y+\upsilon)\right] + \delta_t y - a_{yl} y - \sum_{i=1}^{l-1} p_i (w_{\tau i} - b_{\tau i})$$

where $w_{\tau i} \equiv \sum_{t \neq \tau} \omega_{ti}$, $a_{\tau i} \equiv \sum_{t \neq \tau} \alpha_{ti}$, $b_{\tau i} \equiv \sum_{t \neq \tau} \beta_{ti}$, $g_\tau \equiv \sum_{t \neq \tau} \gamma_t$, $d_\tau \equiv \sum_{t \neq \tau} \delta_t$, and $a_\tau \equiv \sum_{i=1}^{l-1} a_{\tau i}$. Define

$$z_{\tau i}(p, s/s_\tau) \equiv (w_{\tau i} - b_{\tau i}) - (a_{\tau i}/p_i) \quad (i = 1, \cdots, l-1), \text{ and}$$
$$z_{\tau l}(p, s/s_\tau) \equiv a_\tau - \sum_{i=1}^{l-1} p_i (w_{\tau i} - b_{\tau i})$$

Then we have that

$$x_{\tau i}(p, y, a_y, s/s_\tau) = \omega_{\tau i} - a_{yi} y + z_{\tau i}(p, s/s_\tau) \qquad (i = 1, \cdots, l-1), \text{ and}$$

$$x_{\tau l}(p, y, a_y, s/s_\tau) \equiv \omega_{\tau l} - a_{yl} y + \left[g_\tau y/(y+\upsilon)\right] + d_\tau y + z_{\tau l}(p, s/s_\tau),$$

or

$$x_\tau(p, y, a_y, s/s_\tau) \equiv \omega_\tau - a_y y + [d_\tau y + g_\tau y/(y+\upsilon)]e_l + z_\tau(p, s/s_\tau)$$

with e_l denoting the l-th unit vector. Consider the following set-valued mapping

$$Z_\tau(s/s_\tau) \equiv \left\{ z_\tau \in R^l \,\middle|\, \left(\exists p \in P^l\right)\left(z_\tau \leq z_\tau(p, s/s_\tau)\right) \right\}$$

When $a_{\tau i}(w_{\tau i} - b_{\tau i}) > 0$ ($i = 1, \cdots, l-1$), then the above can be rewritten as follows.

$$Z_\tau(s/s_\tau) = \left\{ z'_\tau \in R^l \,\middle|\, z'_\tau \leq z_\tau, z_{\tau l} = a_\tau - \sum_{i=1}^{l-1} a_{\tau i}(w_{\tau i} - b_{\tau i}) / \left\{(w_{\tau i} - b_{\tau i}) - z_{\tau i}\right\} \right\}.$$

The above set-valued mapping has the following properties.

Lemma 1. For each $\tau \in T$ and for each s/s_τ, (i) $Z_\tau(s/s_\tau)$ is a convex set bounded from above, and (ii) if $a_{\tau i}(w_{\tau i} - b_{\tau i}) > 0$ $(i = 1, \cdots, l-1)$, then $0_l \in Z_\tau(s/s_\tau)$ and $Z_\tau(s/s_\tau)$ is strictly convex.

The residual consumption map for agent τ is defined by

$$X_\tau(a_y, s/s_\tau) \equiv \left\{ (x_\tau, y) \in R_+^l \times R_+ \,\middle|\, x_\tau - \omega_\tau + a_y y - [d_\tau + g_\tau y/(y+\upsilon)]e_l \in Z_\tau(s/s_\tau) \right\}.$$

The residual consumption map has properties similar to $Z_\tau(s/s_\tau)$ as follows.

Lemma 2. For each $\tau \in T$ and for each s/s_τ, (i) $X_\tau(a_y, s/s_\tau)$ is a convex set bounded from above, and (ii) if $a_{\tau i}(w_{\tau i} - b_{\tau i}) > 0$ ($i = 1, \cdots, l-1$), then $(\omega_\tau, 0) \in X_\tau(a_y, s/s_\tau)$ and the upper frontier of $X_\tau(a_y, s/s_\tau)$ is strictly convex.

The residual cost share function $q_\tau : R_+ \times R_+ \times \prod_{t \neq \tau} S_t \to R$ for the public good will be defined as follows.

$$q_\tau(y, c, s/s_\tau) \equiv c - \sum_{t \neq \tau} k_t(y, s_t) = c - \sum_{t \neq \tau} \left(\delta_t + \gamma_t /(y+\upsilon) \right) = c - \frac{g_\tau}{y+\upsilon} - d_\tau$$

Lemma 3. Given $(p, y, s/s_\tau) \in P^l \times R_+ \times \prod_{t \neq \tau} S_t$, if $x_\tau = x_\tau(p, y, a_y, s/s_\tau) \gg 0_l$ and $q_\tau = q_\tau(c, y, s/s_\tau) \geq 0$, then there exists $s_\tau \in S_\tau^0$ such that $x_\tau = f_\tau(p, y, s_\tau)$ and $q_\tau = k_\tau(y, s_\tau)$ with $(\alpha_\tau, \beta_\tau) \gg 0_{2(l-1)}$.

Lemma 4. If $(p^*, y^*, x^*, s^*) \in P^l \times R_+ \times R_+^{lT} \times S^0$ is a strategic Lindahl equilibrium with respect to the strategy set S^0, then (p^*, y^*, x^*, s^*) is also a strategic Lindahl equilibrium with respect to the strategy set S.

3. The Structure of Strategic Lindahl Equilibrium Allocations

By Lemma 4 of the previous section, it suffices to restrict the strategy set to S^0 in proving the existence of a strategic Lindahl equilibrium. In a related paper [Otani (2001 a)], we show the following theorem on the existence of equilibrium in which prices and consumption vectors are all positive, i.e., $(p^*, x^*) \in P^l \times P^{lT}$, and hence strategic parameters on private goods are also positive.

Theorem 1. There exists a strategic Lindahl equilibrium $(p^*, y^*, x^*, s^*) \in P^l \times R_+ \times R_+^{lT} \times S^0$ so that $(p^*, x^*) \in P^l \times P^{lT}$ and for every $t \in T$, $(\alpha_t^*, \beta_t^*) \in P^{2(l-1)}$ with $\beta_{ti}^* < \omega_{ti}$ $(i = 1, \cdots, l-1)$.

Based on the above existence theorem, we shall examine the manifold structure of strategic Lindahl equilibrium allocations in this section. We note that in proving the existence of a strategic equilibrium, the parameter δ_t in strategic utility functions is redundant and we set $\delta_t \equiv 0$.

We partition agents into T_1 and T_2. For agent τ in T_1 we have $\gamma_\tau^* > 0$ and for agent τ in T_2 we have $\gamma_\tau^* = 0$. Agents in T_2 accept the level y^* of the public good dictated by other agents with positive cost shares and agents in T_1 are positively contributing to the production of the public good and thus want a level of the public good larger than that desired by other agents. Agents in T_1 will be called unconstrained and agents in T_2 constrained. Let the cost share of agent $\tau \in T_1$ be denoted by q_τ^* which satisfies

$$q_\tau^* \equiv \gamma_\tau^* / (y^* + \upsilon) > 0 .$$

We first slightly modify the strategy s^*. Choose any $(\tilde{\gamma}_\tau, \tilde{\delta}_\tau)$, so that $\tilde{\gamma}_\tau > 0, \tilde{\delta}_\tau > 0$ and

$$q_\tau^* = \tilde{\gamma}_\tau / (y^* + \upsilon) + \tilde{\delta}_\tau .$$

Let $\tilde{s}$ be the strategy in which $\tilde{s}_\tau \equiv (\alpha_\tau^*, \beta_\tau^*, \tilde{\gamma}_\tau, \tilde{\delta}_\tau)$ for $\tau \in T_1$ and $\tilde{s}_\tau = s_\tau^*$ for $\tau \in T_2$.

Then it is clear that $(p^*, y^*, x^*, \tilde{s})$ is also a Lindahl strategic equilibrium. We shall be starting from this equilibrium $\tilde{s}$ in which $\tilde{\delta}_\tau > 0$ and still denote it as s^*. The parameter $\delta_\tau^* \equiv \tilde{\delta}_\tau > 0$ may appear redundant, but the presence of this additional parameter facilitates our computation of the size of strategic Lindahl equilibrium allocations.

Agent τ faces the residual supply (or excess supply) map of private goods $x_\tau(p, y, a_y, s/s_\tau)$ of other agents and tries to find the best consumption x'_τ he/she could secure at a particular price, say p', i.e., $x'_\tau = x_\tau(p', y, a_y, s/s_\tau)$. Then he/she would choose a strategy s'_τ so that $x'_\tau = f_\tau(p', y, s'_\tau)$. If for any pair of (p', x'_τ) we can always find a strategy s'_τ so that $x'_\tau = f_\tau(p', y, s'_\tau)$, then finding an optimum strategy s'_τ reduces to finding an optimum p' maximizing $u_\tau(x_\tau, y)$ subject to $x_\tau = x_\tau(p, y, a_y, s/s_\tau)$. Indeed our strategic demand functions for private goods and Assumption 1 guarantees that this is always possible. Since agents perceive that their strategic behaviors do not affect (a_y^*, c^*) and the production of the public good obeys the constant returns to scale, the supply of the public good is infinitely elastic at the price $q^* = c^*$. Moreover if agent τ is in T_1, he/she can choose $(\gamma_\tau, \delta_\tau)$ for any desired level of the public good. Thus we can consider the following maximization problem for agent τ in T_1

$$\max_{(p,y)} u_\tau[x_\tau(p, y, a_y^*, s^*/s_\tau), y]$$

and for agent τ in T_2

$$\max_{p} u_\tau[x_\tau(p, y^*, a_y^*, s^*/s_\tau), y^*]$$

First order conditions applicable to these maximization problems are as follows:

For every $\tau \in T$,

$$[\partial u_\tau(x_\tau^*, y^*)/\partial x_\tau][\partial x_\tau(p^*, y^*, a_y^*, s^*/s_\tau)/\partial p] = 0'_{l-1},$$

and for agent τ in T_1,

$$[\partial u_\tau(x_\tau^*, y^*)/\partial x_\tau][\partial x_\tau(p^*, y^*, a_y^*, s^*/s_\tau)/\partial y] + [\partial u_\tau(x_\tau^*, y^*)/\partial y] = 0 .$$

In view of residual demand functions we have that

$$\partial x_{\tau i}\big/\partial p_j = 0 \quad (i \neq j; i, j = 1, \ldots, l-1) ,$$

$$\partial x_{\tau i}\big/\partial p_i = a_{\tau i}\big/ p_i^2 \quad (i = 1, \ldots, l-1) ,$$

$$\partial x_{\tau l}\big/\partial p_i = w_{\tau i} - b_{\tau i} ,$$

$$\partial x_{\tau i}\big/\partial y = -a_{yi} ,$$

$$\partial x_{\tau l}\big/\partial y = -a_{yl} + d_\tau + g_\tau\, \upsilon\big/(y+\upsilon)^2$$

where $x_\tau = x_\tau(p, y, a_y, s/s_\tau)$. Thus the above system of equations can be reduced to the following: for every $i = 1, \cdots, l-1$, and for every $\tau \in T$

$$[\partial u_\tau/\partial x_{\tau i}]\left(a_{\tau i}\big/ p_i^2\right) - [\partial u_\tau/\partial x_{\tau l}](w_{\tau i} - b_{\tau i}) = 0 ,$$

and for every $\tau \in T_1$

$$-\sum_{i=1}^{l}[\partial u_\tau/\partial x_{\tau i}]a_{yi} + [\partial u_\tau/\partial x_{\tau l}][d_\tau + g_\tau \upsilon\big/(y+\upsilon)^2] + [\partial u_\tau/\partial y] = 0$$

which must be evaluated at (p^*, y^*, x^*, s^*). Consider the following relations

$$x_T - \omega_T = \sum_{t \neq T}(\omega_t - x_t) - a_y(p)y$$

$$\beta_{ti} = x_{ti} - (\alpha_{ti}/p_i) \quad (i = 1, \cdots, l-1; t \in T)$$

$$q_t = c - \left(d_\tau + g_\tau\big/(y+\upsilon)\right) \text{ or } d_\tau = (c - q_\tau) - g_\tau\big/(y+\upsilon) \quad (t \in T_1) ,$$

$$q_{T_1} = c(p) - \sum_{t \in T_1, t \neq T_1} q_t \text{ , and}$$

$$w_{\tau i} - b_{\tau i} = \sum_{t \neq \tau}(\omega_{ti} - x_{ti} + \alpha_{ti}\big/p_i) = x_{\tau i} - \omega_{\tau i} + a_{yi}y + \left(\sum_{t \neq \tau}\alpha_{ti}\right)\Big/ p_i$$

$$(i = 1, \cdots, l-1; \tau \in T) .$$

Let us define

$$\tilde{p} \equiv (p_1, \ldots, p_{l-1}) ,$$

$$\tilde{q} \equiv (q_1, \ldots, q_{T_1-1}) ,$$

$$\tilde{x}_{T-1} \equiv (x_t)_{t=1}^{T-1} \in P^{(T-1)l},$$

$$\tilde{s} \equiv \left((\alpha_t)_{t=1}^{T}, (\gamma_t)_{t=1}^{T_1}\right) \in P^{(l-1)T} \times R_+^{T_1},$$

$$r_{ti}(\rho) \equiv \partial u_t(x_t, y) / \partial x_{ti} \quad (i = 1, \dots, l-1; t \in T), \text{ and}$$

$$r_{ty}(\rho) \equiv \partial u_t(x_t y) / \partial y \quad (t \in T).$$

Then the left-hand side of the first-order conditions can be expressed as the following system of functions of $(\tilde{s}, \tilde{p}, \tilde{x}_{T-1}, y)$ as follows.

$$H_{\tau i}(\tilde{s}, \tilde{q}, \tilde{p}, \tilde{x}_{T-1}, y) \equiv \left\{\left(r_{\tau i} / p_i^2\right) - \left(r_{\tau l} / p_i\right)\right\}\left(\sum\nolimits_{t \neq \tau} \alpha_{ti}\right) - r_{\tau l}(x_{\tau i} - \omega_{\tau i} + a_{yi} y)$$

$((i = 1, \dots l-1; \tau \neq T)$,

$$H_{Ti}(\tilde{s}, \tilde{q}, \tilde{p}, \tilde{x}_{T-1}, y) \equiv \left\{\left(r_{Ti} / p_i^2\right) - \left(r_{Tl} / p_i\right)\right\}\left(\sum\nolimits_{t \neq T} \alpha_{ti}\right) - r_{Tl}\left\{\sum\nolimits_{t \neq T} (\omega_{ti} - x_{ti})\right\} \quad (i = 1, \dots, l-1)$$

$$J_\tau(\tilde{s}, \tilde{q}, \tilde{p}, \tilde{x}_{T-1}, y) \equiv -\sum\nolimits_{i=1}^{l} r_{\tau i} a_{yi} + r_{\tau l}\left(\left(c(p) - q_\tau\right) - \left(y/(y+\upsilon)^2\right)\sum\nolimits_{t \neq \tau, t \in T_1} \gamma_t\right) + r_{\tau y}$$

$$(\tau \in T_1, \tau \neq T_1),$$

$$J_{T_1}(\tilde{s}, \tilde{q}, \tilde{p}, \tilde{x}_{T-1}, y) \equiv -\sum r_{T_1 i} a_{yi} + r_{T_1 l}\left\{\sum\nolimits_{t \neq T_1} q_t - g_{T_1}\, y\big/(y+\upsilon)^2\right\} + r_{T_1 y},$$

$$K_\tau(\tilde{s}, \tilde{q}, \tilde{p}, \tilde{x}_{T-1}, y) \equiv p \cdot (x_\tau - \omega_\tau) + q_\tau y \qquad (\tau = 1, \dots, T_1 - 1),$$

$$K_\tau(\tilde{s}, \tilde{q}, \tilde{p}, \tilde{x}_{T-1}, y) \equiv p \cdot (x_t - \omega_t) \quad (\tau \in T_2).$$

If we define $H_\tau' \equiv (H_{\tau 1}, \cdots, H_{\tau(l-1)})'$, $H' \equiv (H_\tau')_{\tau=1}^{T}$, $J' \equiv (J_1, \cdots J_{T_1})$, $(K^1)' \equiv (K_1, \dots, K_{T_1 - 1})$,

$(K^2)' \equiv (K_{T_1+1}, \dots, K_T)$, $K' \equiv \left((K^1)', (K^2)'\right)$ and $F' \equiv (H', J', K')$. Then we have that

$$F(\tilde{s}^*, \tilde{q}^*, \tilde{p}^*, \tilde{x}_{T-1}^*, y^*) = 0_{lT+T_1}$$

where (p^*, y^*, x^*, s^*) is a strategic Lindahl equilibrium and $q_\tau^* = k_\tau(p^*, y^*, s_\tau^*)$.

It may be noted that the budget condition for agent T_1 is not in the system of equations because of the Walras law, but this presumes that $T_1 \geq 1$ or $y^* > 0$. There are $(l-1)T$ equations in H, T_1 equations in J, $(T-1)$ equations in K. Thus the total number of equations in the above system F is $(l-1)T + T_1 + T - 1 = lT + T_1 - 1$. Our main interest is to find the degree of freedom in the set of consumption allocations $(\tilde{x}_{T-1}, y)$

whose dimension is $l(T-1)+1$. The number of variables in $(\tilde{s},\tilde{q},\tilde{p})$ is $(l-1)T+T_1+(T_1-1)+(l-1)=lT+T_1-1-(T-T_1)+(l-1)$. Thus if the number of variables in $(\tilde{s},\tilde{q},\tilde{p})$ is greater than or equal to that of equations in F, i.e., if $l-1\geq T-T_1=T_2$, then $(\tilde{x}_{T-1},y)$ and some of variables in $(\tilde{s},\tilde{q},\tilde{p})$ will possibly be free. If otherwise, i.e., $l-1<T_2$, then (T_2-l+1) variables in $(\tilde{x}_{T-1},y)$ would be bound and the degree of freedom in $(\tilde{x}_{T-1},y)$ would possibly be reduced to $lT-T_2=(l-1)T+T_1$. But of course these computations of degrees of freedom are subject to the rank condition of the Jacobian of F which will be examined below.

First note that the mapping H is independent of γ, the mapping J is independent of α, the mapping K^1 is independent of (α,γ) and the mapping K^2 is independent of $(\alpha,\gamma,\tilde{q})$. Derivatives of H_τ with respect to α_t, the derivatives of J_τ with respect to γ_t and the derivatives of K_τ with respect to $\tilde{p}$ and x_t are given as follows:

$$\partial H_\tau / \partial \alpha_\tau = 0_{(l-1)\times(l-1)},$$

$$\partial H_\tau / \partial \alpha_t = D\left[(r_{\tau i}/p_i^2)-(r_{\tau l}/p_i)\right]_{i=1}^{l-1} \quad (t\neq\tau),$$

$$\partial J_\tau / \partial \gamma_\tau = 0,$$

$$\partial J_\tau / \partial \gamma_t = \left(\upsilon/(y+\upsilon)^2\right) r_{\tau l} \quad (t\neq\tau),$$

$$\partial K_\tau / \partial \tilde{p} = (\tilde{x}_\tau - \tilde{\omega}_\tau)' \quad (\tau=1,\cdots,T-1),$$

$$\partial K^1 \big/ \partial \tilde{q} = D\left[y\right]_{i=1}^{T_1-1},$$

$$\partial K_\tau \big/ \partial x_\tau = p' = (\tilde{p}',1) \qquad (\tau=1,\cdots,T-1),$$
$$\partial K_\tau \big/ \partial x_t = 0_l' \qquad (\tau=1,\ldots,T, t\neq\tau)$$

where $D\left[a_i\right]_{i=1}^{l-1}$ denotes a diagonal $(l-1)\times(l-1)$-matrix with a_i as the i-th diagonal element. Since both $\partial H_\tau/\partial\alpha_t$ and $\partial J_\tau/\partial\gamma_t$ $(t\neq\tau)$ are independent of t, we may write

$$\partial H_\tau / \partial \alpha_t \equiv D_{\alpha\tau} \ \text{ and } \ \partial J_\tau / \partial \gamma_t \equiv D_{\gamma\tau} \quad (t\neq\tau).$$

First suppose that $T_2 \leq l-1$. Generically excess demand vectors $(\tilde{x}_\tau - \tilde{\omega}_\tau)$ for $\tau \in T_2$ are linearly independent. Thus generically the $T_2\times(l-1)$-matrix $\partial K^2\big/\partial\tilde{p}$ contains a $T_2\times T_2$-submatrix with the full rank denoted by $\tilde{K}^2$ and we let the $T_2\times 1$ price vector corresponding to $\tilde{K}^2$ be $\tilde{p}^1$. Let $\chi\equiv(\tilde{s},\tilde{q},\tilde{p}^1)$, $\tilde{p}\equiv(\tilde{p}^1,\tilde{p}^2)$, and

$\rho \equiv (\tilde{p}^2, \tilde{x}_{T-1}, y)$. Note that the dimension of χ is $lT + T_1 - 1$ which coincides with the number of equations in the system F. Then the Jacobian of F with respect to χ is given as follows:

$$\partial F / \partial \chi \equiv \begin{bmatrix} \partial H / \partial \alpha & * & * & * \\ 0_{T_1 \times (l-1)T} & \partial J / \partial \gamma & * & * \\ 0_{(T_1-1)\times(l-1)T} & 0_{(T_1-1)\times T_1} & D[y]_{i=1}^{T_1-1} & * \\ 0_{T_2 \times (l-1)T} & 0_{T_2 \times T_1} & 0_{T_2 \times (T_1-1)} & \tilde{K}^2 \end{bmatrix}$$

where

$$\partial H / \partial \alpha = \begin{bmatrix} 0_{(l-1)\times(l-1)} & D_{\alpha 1} & \cdots & D_{\alpha 1} \\ D_{\alpha 2} & 0_{(l-1)\times(l-1)} & \cdots & D_{\alpha 2} \\ \vdots & \cdots & \ddots & \vdots \\ D_{\alpha T} & D_{\alpha T} & \cdots & 0_{(l-1)\times(l-1)} \end{bmatrix}, \text{ and}$$

$$\partial J / \partial \gamma = \begin{bmatrix} 0 & D_{\gamma 1} & \cdots & D_{\gamma 1} \\ D_{\gamma 2} & 0 & \cdots & D_{\gamma 2} \\ \vdots & \cdots & \ddots & \vdots \\ D_{\gamma T_1} & D_{\gamma T_1} & \cdots & 0 \end{bmatrix}.$$

The following lemma on determinants is easy to prove.

Lemma 5. Let A_i be an $(n \times n)$ real matrix for $i = 1, \cdots, m$. Then

$$\det \begin{bmatrix} 0_{n\times n} & A_1 & \cdots & A_1 \\ A_2 & 0_{n\times n} & \cdots & A_2 \\ \vdots & \cdots & \ddots & \vdots \\ A_m & A_m & \cdots & 0_{n\times n} \end{bmatrix} = (-1)^{n(m-1)}(m-1)^n \prod_{i=1}^{m} \det(A_i)$$

where $\det(A)$ denotes the determinant of the matrix A.

If we apply Lemma 5, then determinants of $\partial F / \partial \chi$, $\partial H / \partial \alpha$, and $\partial J / \partial \gamma$ can be written as in the following lemma.

Lemma 6. (i) For the case of $T_2 \le l - 1$, $\partial F / \partial \chi = \det(\partial H / \partial \alpha)$

$$\times \det\left(\partial J / \partial \gamma\right) \times y^{T_1 - 1} \times \det\left(\tilde{K}^2\right);$$

(ii) $\det\left(\partial H / \partial \alpha\right) = (-1)^{(l-1)(T-1)}(T-1)^{l-1} \prod_{t=1}^{T} \det\left(D_{\alpha t}\right);$

(iii) $\det\left(\partial J / \partial \gamma\right) = (-1)^{T_1 - 1}(T_1 - 1) \prod_{t=1}^{T_1} \det\left(D_{\gamma t}\right).$

We next suppose that $(l-1) \le T_2$ which would perhaps be a more interesting case. We divide agents in T_2 into $\overline{T}_2$ of first $(l-1)$ agents and $\overline{\overline{T}}_2$ of the rest of $T_2-(l-1)$ agents. Correspondingly we partition the matrix $\partial K^2 / \partial \tilde{p}$ as follows:

$$\partial K^2 / \partial \tilde{p} \equiv \begin{bmatrix} \overline{K}^2 \\ \overline{\overline{K}}^2 \end{bmatrix}$$

where

$$\overline{K}^2 \equiv \left[\partial K_t^2 / \partial \tilde{p}\right]_{t \in \overline{T}_2}, \overline{\overline{K}}^2 \equiv \left[\partial K_t^2 / \partial \tilde{p}\right]_{t \in \overline{\overline{T}}_2}.$$

Note that $\partial K_\tau^2 / \partial x_{\tau l} = 1$ and $\partial K_t^2 / \partial x_{\tau l} = 0$ for $t \ne \tau$. Let us define $\overline{\overline{x}}_l = (x_{tl})_{t \in \overline{\overline{T}}_2}$ and the rest of private goods in the vector $\tilde{x}_{T-1}$ be denoted by $\tilde{x}_{T-1} / \overline{\overline{x}}_l$ Then

$$\partial K^2 / \partial \overline{\overline{x}}_l = \begin{bmatrix} 0_{(l-1) \times \overline{\overline{T}}_2} \\ I_{\overline{\overline{T}}_2} \end{bmatrix}.$$

We then define the following $T_2 \times T_2$ matrix denoted by $\tilde{\tilde{K}}_2$:

$$\tilde{\tilde{K}}_2 \equiv \left[\partial K^2 / \partial \tilde{p} \quad \partial K^2 / \partial \overline{\overline{x}}_l\right] = \begin{bmatrix} \overline{K}^2 & 0_{(l-1) \times \overline{\overline{T}}_2} \\ \overline{\overline{K}}_2 & I_{\overline{\overline{T}}_2} \end{bmatrix}.$$

For the case of $(l-1) \le T_2$, we let $\chi \equiv (\tilde{s}, \tilde{q}, \tilde{p}, \overline{\overline{x}}_l)$, and $\rho \equiv (\tilde{x}_{T-1} / \overline{\overline{x}}_l, y)$. Note that the dimension of χ is again $lT + T_1 - 1$. Then the Jacobian of F with respect to χ is given as follows:

$$\partial F/\partial\chi \equiv \begin{bmatrix} \partial H/\partial\alpha & * & * & * \\ 0_{T_1\times(l-1)T} & \partial J/\partial\gamma & * & * \\ 0_{(T_1-1)\times(l-1)T} & 0_{(T_1-1)\times T_1} & D[y]_{i=1}^{T_1-1} & * \\ 0_{T_2\times(l-1)T} & 0_{T_2\times T_1} & 0_{T_2\times(T_1-1)} & \tilde{\tilde{K}}^2 \end{bmatrix}$$

Lemma 7. (i) For the case of $(l-1)\le T_2$, $\partial F/\partial\chi = \det\left(\partial H/\partial\alpha\right)$

$$\times\det\left(\partial J/\partial\gamma\right)\times y^{T_1-1}\times\det\left(\bar{K}^2\right).$$

Assumption 2. (Regularity) (i) At a strategic equilibrium, either $rank\left(\tilde{K}_2\right) = T_2$ when $T_2 \le l-1$ or $rank\left(\bar{K}_2\right) = l-1$ when $(l-1)\le T_2$; (ii) for every $\tau\in T_2$,

$$[\partial u_\tau(x_\tau^*, y^*)/\partial x_\tau][\partial x_\tau(p^*, y^*, a_y^*, s^*/s_\tau)/\partial y] + [\partial u_\tau(x_\tau^*, y^*)/\partial y] < 0$$

holds at a strategic equilibrium; and (iii) at the strategic Lindahl equilibrium $(p^* y^*, x^*, s^*)$, no agent is Walrasian in any private good market in the sense that $r_{\tau i} - r_{\tau l}p_i \ne 0$ for every $\tau\in T$ and for every $i = 1,\cdots,l-1$.

We note that $\partial K_\tau/\partial\tilde{p} = (\tilde{x}_\tau - \tilde{\omega}_\tau)'$. Thus $\partial K^2/\partial\tilde{p}$ is a $T_2\times(l-1)$-matrix consisting of excess demand vectors for agents in T_2. When $T_2\le l-1$, $\partial K^2/\partial\tilde{p}$ generically has an $T_2\times T_2$-submatrix $\tilde{K}^2$ with the full rank T_2. Similarly when $(l-1)\le T_2$, $\partial K^2/\partial\tilde{p}$ generically contains an $(l-1)\times(l-1)$-submatrix $\bar{K}^2$ with the full rank $(l-1)$. This is Assumption 2 (i).

Assumption 2 (ii) implies that in a neighborhood of a strategic equilibrium, agents in T_2 remain to be in T_2. Also note that if $r_{\tau i} - r_{\tau l}p_i = 0$, then the marginal rate of substitution between good i and good l is equal to p_i indicating the price taking condition. By the first-order conditions, $r_{\tau i} - r_{\tau l}p_i = 0$ implies that $(x_{\tau i} - \omega_{\tau i}) + a_{yi}y = 0$ and thus, at least at the equilibrium, agent τ is self-sufficient in (or not trading) good i. It is clear that Assumptions 2 (ii) and (iii) hold generically.

Under Assumption 2, the Jacobian of F with respect to the variable χ becomes full in both cases, i.e.,

$$rank\left(\partial F(\chi^*, \rho^*)/\partial\chi\right) = lT + T_1 - 1.$$

Therefore by the implicit function theorem, there exists a unique smooth mapping $\chi = \xi(\rho)$ from a neighborhood of ρ^* into a neighborhood of χ^*. Then define a mapping ψ from a neighborhood of (χ^*, ρ^*) into R^M with $M = \#(\chi) + \#(\rho)$ as follows:

$$\psi(\chi, \rho) \equiv (\chi - \xi(\rho), \rho).$$

Clearly the inverse mapping of ψ is given by $\psi^{-1}(\overline{\chi}, \overline{\rho}) = (\overline{\chi} + \xi(\overline{\rho}), \overline{\rho})$ and thus ψ is a local diffeomorphism. Moreover $(\chi, \rho) \in F^{-1}(0_{lT+T_1})$ if and only if $\psi(\chi, \rho) = (0_{\#(\chi)}, \rho)$.

Thus we have shown that $F^{-1}\left(0_{lT+T_1-1}\right)$ is a differentiable submanifold of dimension $\#(\rho)$ with ρ serving as a local coordinate system. We state this as a theorem.

Theorem 2. Under Assumptions 1 and 2, $F^{-1}\left(0_{lT+T_1-1}\right)$ is a differentiable submanifold of dimension $\#(\rho)$ with ρ serving as a local coordinate system.

If any ρ in a neighborhood of ρ^* is given, χ can be determined by $\chi = \xi(\rho)$. Then the pair (χ, ρ) determines a strategic Lindahl equilibrium. Our interest is on the size of strategic Lindahl equilibrium allocations. When $T_2 \le (l-1)$, then $\rho = (\tilde{p}^2, \tilde{x}_{T-1}, y)$ and thus $(\tilde{x}_{T-1}, y)$ will be the free variable in the space of allocations. Therefore the dimension of equilibrium allocations will be $l(T-1)+1$. On the other hand, when $(l-1) \le T_2$, then $\rho \equiv (\tilde{x}_{T-1}/\overline{\overline{x}}_l, y)$. Thus the dimension of equilibrium allocations will be $lT - T_2 = (l-1)T + T_1$.

Proposition 1. Under Assumptions 1 and 2, generically the set of strategic Lindahl equilibrium allocations is a differentiable manifold of dimension $l(T-1)+1$ when $T_2 \le (l-1)$ and $lT - T_2 = (l-1)T + T_1$ when $(l-1) \le T_2$ provided that $T_1 \ge 1$ or $y^* > 0$,.

For the case of a private good economy with l private goods and T agents, we have shown in Otani (1996) that the dimension of indeterminacy in the set of strategic equilibrium allocations is of (i) $l(T-1)$ when $T \le l$ and (ii) $(l-1)T$ when $T \ge l$. Case

(i) of $T \leq l$ for the private good economy is analogous to the case of $T_2 \leq (l-1)$ in the public good economy resulting in the dimension of indeterminacy of $l(T-1)+1$ increasing by the number of the public good compared to the private good economy. Case (ii) of $T \geq l$ for the private good economy corresponds to the case of $(l-1) \leq T_2$. In this case, the dimension of indeterminacy becomes $lT - T_2 = (l-1)T + T_1$ increasing by the number T_1 of contributing agents. Thus when all agents are free riding or not contributing for a positive level of the public good, the economy degenerates to a private goods economy with no production of the public good and the dimension of indeterminacy coincides with that of private goods economies.

Let us consider the case of a sequence of replica economies in which $T = nT^*$ with T^* denoting the number of agent types and n denoting the number of replicas. Similarly we write $T_i = nT_i^*$. Then the average dimension per replica for the private good economy in Otani (1996) becomes (i) $lT^* - l/n$ when $T \leq l$ and (ii) $(l-1)T^*$ when $T \geq l$.

On the other hand, the average dimension per replica for our public good economy when $T_2 \leq (l-1)$ becomes

$$\{l(T-1)+1\}/n = lT^* - [(l-1)/n].$$

The above converges to T^*l as $n \to \infty$. Therefore the set of equilibrium allocations stays large as the economy gets larger in the sense of $n \to \infty$. But this is the same size of indeterminacy as in the case of the private good economy in Otani (1996) since $lT^* - l/n$ converges to T^*l as $n \to \infty$.

When $(l-1) \leq T_2$, the average dimension per replica for our public good economy becomes

$$(l-1)T^* + T_1^*.$$

Thus again the set of equilibrium allocations stays large as the economy gets larger and the dimension of indeterminacy increases by the number of contributing agent types compared with the private good economy.

It may be noted that in a large economy, the latter case of $(l-1) \leq T_2$ will tend to prevail since $(l-1) \leq T_2$ can be rewritten as $(l-1)/n \leq T_2^*$ and $(l-1)/n$ converges to 0 as $n \to \infty$. Thus we can conclude that the dimension of indeterminacy for the public good economy will stay large for a large economy and will be larger by the number of contributing agent types compared with the private good economy.

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